

GEOMETRIC STRUCTURES ASSOCIATED WITH A SIMPLE CARTAN 3-FORM

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ABSTRACT. We introduce the notion of a manifold admitting a simple compact Cartan 3-form ω^3 . We study algebraic types of such manifolds specializing on those having skew-symmetric torsion, or those associated with a closed or coclosed 3-form ω^3 . We prove the existence of an algebra of multi-symplectic forms ϕ^l on these manifolds. Cohomology groups associated with complexes of differential forms on M^n in presence of such a closed multi-symplectic form ϕ^l and their relations with the de Rham cohomologies of M are investigated. We show rigidity of a class of strongly associative (resp. strongly coassociative) submanifolds. We include an appendix describing all connected simply connected complete Riemannian manifolds admitting a parallel 3-form.

CONTENTS

1. Introduction	1
2. Cartan 3-form $\omega_{\mathfrak{g}}$ and its stabilizer group	3
3. Examples of manifolds provided with a 3-form ω^3 of type $\omega_{\mathfrak{g}}$	5
4. Algebraic types of manifolds provided with a 3-form ω^3 of type $\omega_{\mathfrak{g}}$	7
5. Cohomology theories for $Aut^+(\mathfrak{g})$ -manifolds equipped with a quasi-closed form of type ϕ_0^l	14
6. Special submanifolds in $Aut(\mathfrak{g})$ -manifolds	24
7. Final remarks	28
8. Appendix. Riemannian manifolds equipped with a parallel 3-form	28
Acknowledgement	31
References	31

AMSC2010: 53C10, 53C2, 53C38.

Key words: Cartan 3-form, multi-symplectic form, G -structure

1. INTRODUCTION

Let G be a Lie subgroup of $O(n)$. We want to characterize a class of “natural” G -structures on Riemannian manifolds M^n . First, we would like

Date: January 13, 2013.

to see G in the list of possible holonomy groups of Riemannian manifolds M^n . Second, we also like to characterize G as the stabilizer group of some exterior k -form on \mathbb{R}^n (as it is the case with most of special holonomy groups of Riemannian manifolds, see [2, table 1, chapter 10]). Note that the holonomy group of a Riemannian manifold M^n lies in such a “natural” group $G \subset O(n)$ only if M^n admits a parallel k -form ϕ^k , which makes M^n a calibrated manifold. A careful analysis shows that a connected simply connected complete Riemannian manifold M^n admits a parallel 3-form ϕ^3 , if and only if (M^n, ϕ^3) is a product of basic Riemannian manifolds (M^i, ϕ_i^3) , where either $\phi_i^3 = 0$, or M^i is flat and ϕ_i^3 is a parallel form, or ϕ_i^3 is one of stable 3-forms in dimensions 6, 7, 8, or ϕ^3 is a wedge product of a Kähler 2-form with a 1-form, or ϕ^3 is a Cartan 3-form associated with a simple compact Lie group, see Theorem 8.1 for a precise formulation. This motivates us to study geometry associated with a Cartan 3-form. It turns out that these manifolds possess very rich geometric structures, arising from the cohomological structure of the associated Lie algebra.

Our study can be thought as a continuation of the study initiated by Hitchin of geometries associated with stable 3-forms in dimension 6, 7, 8, [11], [12], which includes the Special Lagrangian (SL) 3-form, the 3-form of G_2 -type, and the Cartan 3-form on $\mathfrak{su}(3)$. On the other hand, our study provide new examples and some structure theorems for the theory of manifolds provided with a closed multi-symplectic form, which has been discovered long time ago in relation with the multi-variate field theory [9], and enjoys its active development nowadays [1], [23].

The plan of our note is as follows. In section 2 we recall the definition of a Cartan 3-form $\omega_{\mathfrak{g}}$ and show that it is multi-symplectic if and only if \mathfrak{g} is semisimple, see Lemma 2.1. We compute the stabilizer of $\omega_{\mathfrak{g}}$ in the case that \mathfrak{g} is a simple Lie algebra over \mathbb{C} or a real form of a simple complex Lie algebra, see Theorem 2.2. In section 3 we present many examples of manifolds provided with a simple Cartan 3-form. In section 4, using the notion of intrinsic torsion, we prove several structure theorems on algebraic types of a manifold provided with a compact simple Cartan form ω^3 , especially on those algebraic types having skew-symmetric torsions, and those associated with a closed or coclosed 3-form ω^3 of type $\omega_{\mathfrak{g}}$, see Proposition 4.6 and Theorem 4.8, Lemma 4.11, Corollary 4.12. We end this section with a theorem describing torsion-free complete $Aut(\mathfrak{g})$ -manifolds, see Theorem 4.13. In section 5 we show the existence of an algebra of nowhere vanishing multi-symplectic forms ϕ^l on an orientable manifold M^n equipped with a compact Cartan 3-form, see Theorem 5.1 and Lemma 5.3. We study cohomology groups associated with a closed multi-symplectic form ϕ^l of the considered type and their relations with the de Rham cohomology groups of M^n , see Theorem 5.9, Example 5.11, Proposition 5.14, Lemma 5.17. In section 6 we study a class of strongly associative (or strongly coassociative) submanifolds in a manifold M^n provided with a compact simple Cartan 3-form and prove their algebraic and geometric rigidity, see Proposition 6.2, Proposition 6.4,

Proposition 6.6, Proposition 6.8, Remark 6.9. In section 7 we discuss some questions for further research. In Appendix we describe simply-connected complete Riemannian manifolds admitting a parallel 3-form.

2. CARTAN 3-FORM $\omega_{\mathfrak{g}}$ AND ITS STABILIZER GROUP

In this section we recall the definition of the Cartan 3-form associated with a semisimple Lie algebra \mathfrak{g} . We show that a Cartan 3-form is multi-symplectic, see Lemma 2.1. We compute the stabilizer group of the Cartan 3-form in the case that \mathfrak{g} is a simple Lie algebra over \mathbb{C} , or a real form of a simple Lie algebra over \mathbb{C} , see Theorem 2.2. We discuss a generalization of Theorem 2.2 in Remark 2.4.3.

Let \mathfrak{g} be a semisimple Lie algebra over \mathbb{C} or over \mathbb{R} . The Cartan 3-form $\omega_{\mathfrak{g}}$ is defined on \mathfrak{g} as follows

$$\omega_{\mathfrak{g}}(X, Y, Z) := \langle X, [Y, Z] \rangle,$$

where \langle, \rangle denotes the Killing bilinear form on \mathfrak{g} .

Let \mathbb{F} be field \mathbb{C} or \mathbb{R} . We recall that a k -form ω on \mathbb{F}^n is *multi-symplectic*, if the linear map

$$(2.1) \quad L_{\omega} : \mathbb{F}^n \rightarrow \Lambda^{k-1}(\mathbb{F}^n)^*, v \mapsto v \lrcorner \omega,$$

is an injective map.

Lemma 2.1. *The Cartan 3-form $\omega_{\mathfrak{g}}$ is multi-symplectic, if and only if \mathfrak{g} is semisimple.*

Proof. Assume that \mathfrak{g} is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Taking into account the non-degeneracy of the Killing bilinear form on \mathfrak{g} , we get immediately the “if” assertion. Now assume that $\omega_{\mathfrak{g}}$ is multi-symplectic. Then the Killing form \langle, \rangle is non-degenerate, since the kernel of the Killing form lies in the kernel of $L_{\omega_{\mathfrak{g}}}$. This proves the “only if” assertion. \square

Next we note that the stabilizer group $Stab(\omega_{\mathfrak{g}})$ of $\omega_{\mathfrak{g}}$ contains the automorphism group $Aut(\mathfrak{g})$ of the Lie algebra \mathfrak{g} .

Theorem 2.2. (cf. [6], [14, Theorem 7]) *1. Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} . Then $Stab(\omega_{\mathfrak{g}}) = Aut(\mathfrak{g}) \times \mathbb{Z}_3$ if $\dim \mathfrak{g} > 3$. If $\dim \mathfrak{g} = 3$ then $Stab(\omega_{\mathfrak{g}}) = SL(\mathfrak{g})$.*

2. Let \mathfrak{g} be a real form of a complex simple Lie algebra over \mathbb{C} . Then $Stab(\omega_{\mathfrak{g}}) = Aut(\mathfrak{g})$, if $\dim \mathfrak{g} \geq 3$. If $\dim \mathfrak{g} = 3$, then $Stab(\omega_{\mathfrak{g}}) = SL(\mathfrak{g})$.

Proof. Kable [14, Theorem 7] showed that the stabilizer group of the Cartan 3-form $\omega_{\mathfrak{g}}$ on a semisimple Lie algebra \mathfrak{g} is a semi-direct product of $Aut(\mathfrak{g})$ with an abelian subgroup $M(\mathfrak{g})$ consisting of elements $g \in GL(\mathfrak{g})$ such that g commutes with the adjoint action of \mathfrak{g} , and $g^3 = 1$. Clearly Theorem 2.2 follows from Kable’s theorem and Shur’s lemma.

We also obtain Theorem 2.2 from the result by Freudenthal, who computed the identity component of the group $Stab(\omega_{\mathfrak{g}})$ [6]. Denote by $\mathcal{N}_{GL(\mathfrak{g})}(\mathfrak{g})$

the normalizer of \mathfrak{g} in $GL(\mathfrak{g})$, and by $\mathcal{Z}_{GL(\mathfrak{g})}(\mathfrak{g})$ the centralizer of \mathfrak{g} in $GL(\mathfrak{g})$. We observe that there is a monomorphism

$$(2.2) \quad \mathcal{N}_{GL(\mathfrak{g})}(\mathfrak{g})/\mathcal{Z}_{GL(\mathfrak{g})}(\mathfrak{g}) \rightarrow Aut(\mathfrak{g}).$$

By Freudenthal's theorem the group $Stab(\omega_{\mathfrak{g}})$ is a subgroup of $\mathcal{N}_{GL(\mathfrak{g})}(\mathfrak{g})$. Since the adjoint representation of \mathfrak{g} is irreducible, Shur's lemma implies that $\mathcal{Z}_{GL(\mathfrak{g})}(\mathfrak{g})$ is the center $\mathcal{Z}(GL(\mathfrak{g}))$ of $GL(\mathfrak{g})$. Using (2.2) we get the inclusion $Stab(\omega_{\mathfrak{g}}) \subset Aut(\mathfrak{g}) \times \mathbb{Z}_3$, where $\mathbb{Z}_3 = \mathcal{Z}(GL(\mathfrak{g})) \cap Stab(\omega_{\mathfrak{g}})$, if \mathfrak{g} is a complex simple Lie algebra of dimension at least 8. We also get the inclusion $Stab(\omega_{\mathfrak{g}}) \subset Aut(\mathfrak{g})$, if \mathfrak{g} is a real form of a complex simple Lie algebra. Now Theorem 2.2 follows immediately, observing that $Aut(\mathfrak{g}) \subset Stab(\omega_{\mathfrak{g}})$. \square

Remark 2.3. In his famous paper [4] Dynkin explained how one can apply his result to determine the Lie algebra of the stabilizer group of a G -invariant differential form, where G is a simple Lie group. Dynkin's idea and his classification result in [4] form a main ingredient of Kable's proof (and our unpublished proof) of Theorem 2.2. We thank Dmitri Panyushev for showing us Kable's paper and Freudenthal's results after a proof of Theorem 2.2 is obtained.

Remark 2.4. 1. Theorem 2.2.2 is a generalization of our theorem in [21] for case $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and its real forms, which has been proved by a different method.

2. Let $Ad(\mathfrak{g})$ denote the adjoint group of a Lie algebra \mathfrak{g} . If \mathfrak{g} is a complex simple Lie algebra or a compact form of a complex simple Lie algebra, then $Aut(\mathfrak{g})/Ad(\mathfrak{g})$ is isomorphic to the automorphism group $Aut(D(\mathfrak{g}))$ of the Dynkin diagram $D(\mathfrak{g})$ of $\mathfrak{g}^{\mathbb{C}}$, [10, Theorem IX.5.4, Theorem IX.5.5, Theorem X.3.29]. It is well-known that $Aut(\mathfrak{so}(8)) = \Sigma_3$ -the permutation group on three letters, $Aut(D(\mathfrak{g})) = \mathbb{Z}_2$, if $\mathfrak{g} = \mathfrak{su}_n$ or $\mathfrak{g} = \mathfrak{so}(2n)$, $n \neq 4$, or $\mathfrak{g} = E_6$. In other cases $Aut(D(\mathfrak{g})) = Id$.

3. It follows from Dynkin results [4, Theorem 2.1, Theorem 2.2] that if \mathfrak{g} is a simple complex Lie algebra over \mathbb{C} or over \mathbb{R} such that $\mathfrak{g} \neq \mathfrak{sp}(n, \mathbb{C})$ and $\mathfrak{g} \neq \mathfrak{so}(n, \mathbb{C})$, then the Lie algebra of the stabilizer group of any $Ad(\mathfrak{g})$ -invariant form ϕ^l on \mathfrak{g} coincides with \mathfrak{g} . We conjecture that this assertion also holds for $\mathfrak{g} = \mathfrak{sp}(n)$ or $\mathfrak{g} = \mathfrak{so}(n)$. If the Lie algebra of the stabilizer group of ϕ^l is \mathfrak{g} we can use the same method in the proof of Theorem 2.2 to find the stabilizer group of ϕ^l .

4. Let \mathfrak{g} be a compact simple Lie group. The algebra of $Ad(\mathfrak{g})$ -invariant forms on \mathfrak{g} is equal to the algebra of de Rham cohomologies of the group $Ad(\mathfrak{g})$, which is well-known. The computation of real cohomology of compact Lie groups started by E. Cartan in 1929 was completed for classical groups by R. Brauer and by L. Pontryagin in 1935. H. Hopf and H. Samelson showed in 1941 that $H^*(G, \mathbb{R})$ is an exterior algebra on generators of degrees $2d_1 - 1, \dots, 2d_r - 1$, where $l = rank G$. But only in 1949 C.T. Yen managed to compute the d_i for all exceptional groups, case by case. On the other hand, C. Chevalley and J. Leray showed that the d_i are nothing else but the

degrees of the basic invariants over of the Weyl group of G . Nowadays this is common knowledge [25].

Table 2.4.4.1 (compiled from [13, App.A, Table 6.1.4, p.1742]). The ring $H^*(G, \mathbb{R})$ of a compact simple Lie group G is generated by primitive elements x_i, y_i of degree i as follows.

- 3.1. $H^*(SU(n), \mathbb{R}) = \Lambda(x_3, x_5, \dots, x_{2n-1})$.
- 3.2. $H^*(SO(2n+1), \mathbb{R}) = \Lambda(x_3, x_7, \dots, x_{4n-1})$.
- 3.3. $H^*(SO(2n), \mathbb{R}) = \Lambda(x_3, x_7, \dots, x_{4n-5}, y_{2n-1})$.
- 3.4. $H^*(Sp(n), \mathbb{R}) = \Lambda(x_3, x_7, \dots, x_{4n-1})$.
- 3.5. $H^*(G_2, \mathbb{R}) = \Lambda(x_3, x_{11})$.
- 3.6. $H^*(F_4, \mathbb{R}) = \Lambda(x_3, x_{11}, x_{15}, x_{23})$.
- 3.7. $H^*(E_6, \mathbb{R}) = \Lambda(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23})$.
- 3.8. $H^*(E_7, \mathbb{R}) = \Lambda(x_3, x_{11}, x_{15}, x_{19}, x_{23}, x_{27}, x_{35})$.
- 3.9. $H^*(E_8, \mathbb{R}) = \Lambda(x_3, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}, x_{59})$.

5. If \mathfrak{g} is a complex simple Lie algebra, and \mathfrak{g}_0 is its real form, then any $Ad(\mathfrak{g}_0)$ -invariant form ϕ^l on \mathfrak{g}_0 extends to an $Ad(\mathfrak{g})$ -invariant invariant form on \mathfrak{g} .

3. EXAMPLES OF MANIFOLDS PROVIDED WITH A 3-FORM ω^3 OF TYPE $\omega_{\mathfrak{g}}$

In this section we assume that \mathfrak{g} is a complex simple Lie algebra of dimension $n \geq 8$ or a real form of such a Lie algebra. We introduce the notion of a differential 3-form ω^3 of type $\omega_{\mathfrak{g}}$, see Definition 3.1. We show some examples of manifolds provided with a differential 3-form ω^3 of type $\omega_{\mathfrak{g}}$, and we discuss some possible ways to construct such manifolds, see Examples (3.2) - (3.6).

Definition 3.1. A differential 3-form ω^3 on a manifold M^n is said of *type* $\omega_{\mathfrak{g}}$, if at every point $x \in M^n$ the 3-form $\omega^3(x)$ is equivalent to the Cartan form $\omega_{\mathfrak{g}}$ on \mathfrak{g} , i.e. any linear isomorphism from \mathfrak{g} to $T_x M^n$ sends $\omega^3(x)$ to a 3-form which belongs to the $GL(\mathfrak{g})$ -orbit of $\omega_{\mathfrak{g}}$. If \mathfrak{g} is a complex Lie algebra we require that M^n possesses a volume form as well as an almost complex structure J and the mentioned above linear isomorphism commutes with the (almost) complex structures on $T_x M^n$ and on \mathfrak{g} .

A differential 3-form ω^3 in Definition 3.1 is also called a (*simple, compact*) *Cartan 3-form*, if no misunderstanding occurs.

By Theorem 2.2.2 the existence of a simple Cartan 3-form on M^n is equivalent to the existence of an $Aut(\mathfrak{g})$ -structure on M^n , if \mathfrak{g} is a real form of a complex simple Lie algebra. Below we show examples and a possible construction of manifolds M^n admitting a 3-form of type $\omega_{\mathfrak{g}}$, where $n \geq 8$. (Note that any 3-manifold is parallelizable, so it admits a 3-form of type $\omega_{su(2) \cdot}$)

Example 3.2. Any parallelizable manifold M^n is provided with a simple Cartan 3-form, if $n = \dim_{\mathbb{R}} \mathfrak{g}$. For example any simple Lie group G^n is parallelizable, the manifold $S^{n_1} \times \dots \times S^{n_r}$, $r \geq 2$, $\sum n_i = n$, is parallelizable, if at least one of the n_i is odd [15].

Example 3.3. Assume that \mathfrak{g} is a real form of a simple complex Lie algebra. A homogeneous space K/H admits a Cartan 3-form of type $\omega_{\mathfrak{g}}$ if and only if the isotropy action of H factors through the group $Ad(\mathfrak{g}) \subset \mathfrak{sl}(V)$, where $V = T_e K/H$ and $\dim V = \dim \mathfrak{g}$. Let G be a Lie group with Lie algebra \mathfrak{g} , and $\rho : H \rightarrow G$ - an embedding. We define $K := H \times G$, and let $i : H \rightarrow K = H \times G$ have the form $i = (Id \times \rho)$. Note that the isotropy action of H on V factors thorough the group $Ad(\mathfrak{g})$, and this action is reducible unless $\dim H = \dim G$. For given groups H, G the homogeneous spaces $(H \times G)/H$ may have infinitely many distinct homotopy types depending on an embedding $\rho : H \rightarrow G$, for example, see [17] for the case $G = SU(3)$ and $H = U(1)$.

Example 3.4. Assume that a manifold M^n is equipped with a differential 3-form ω^3 of type $\omega_{\mathfrak{g}}$ and a Lie group K acts on M with cohomogeneity 1 preserving the form ω^3 . Assume that \mathfrak{g} is a real form of a simple complex Lie algebra. Then the principal K -orbit on M^n has the form K/H , where the Lie algebra \mathfrak{h} of H is also a sub-algebra of \mathfrak{g} such that the induced adjoint action of \mathfrak{h} on \mathfrak{g} has a trivial component of dimension 1. Conversely, if H is a subgroup of K such that the sum of the isotropy action of H on $T_e(K/H)$ with the trivial action of H on \mathbb{R} is equivalent to the adjoint action of H on \mathfrak{g} via some embedding $\rho : \mathfrak{h} \rightarrow \mathfrak{g}$, then $(K/H) \times \mathbb{R}$ admits a differential 3-form of type $\omega_{\mathfrak{g}}$. In particular, the direct product $(G/S^1) \times S^1$ admits a 3-form of type $\omega_{\mathfrak{g}}$, if G is a Lie group with Lie algebra \mathfrak{g} . For a given G these spaces may have infinitely many distinct homotopy types, see [17]. Another example of a compact cohomogeneity 1 space is manifold $(SO(5)/SU(2)) \times S^1$ admitting a differential 3-form of type $\omega_{\mathfrak{su}(3)}$ as well as a differential 3-form of type $\omega_{\mathfrak{su}(1,2)}$, since the sum of the isotropy representation of $SU(2)$ on $T_e(SO(5)/SU(2))$ with the trivial representation of $SU(2)$ on \mathbb{R} is equivalent to the adjoint action of $SU(2)$ on $\mathfrak{su}(3)$ as well as to the one on $\mathfrak{su}(1,2)$.

Example 3.5. a) Assume that $M^{n(n-1)/2}$ is provided with an $Ad(\mathfrak{so}(n))$ -structure. Then any differentiable fibration $\mathcal{F}^{n(n+1)/2}$ over a Riemannian manifold M^n with fiber $M^{n(n-1)/2}$ admits a Cartan 3-form of type $\omega_{\mathfrak{so}(n+1)}$, since the frame bundle over $\mathcal{F}^{n(n+1)/2}$ admits a reduction to the subgroup $Ad(\mathfrak{so}(n)) \subset Ad(\mathfrak{so}(n+1)) \subset GL(n(n+1)/2)$. In particular the orthonormal frame bundle $\mathcal{F}(M^n, g)$ of a Riemannian manifold (M^n, g) admits a 3-form of type $\omega_{\mathfrak{so}(n+1)}$.

b) In the same way, a S^1 -bundle over the special unitary frame bundle $\mathcal{F}_{\mathfrak{su}(2)}(M^4)$ of a $K3$ -surface M^4 admits a 3-form of type $\omega_{\mathfrak{su}(3)}$.

Example 3.6. Let \mathfrak{g} be a compact simple Lie subalgebra of a compact Lie algebra $\bar{\mathfrak{g}}$, and \bar{G} - a compact Lie group with Lie algebra $\bar{\mathfrak{g}}$. Let $\dim \mathfrak{g} = k \geq 8$. Denote by $\mathcal{D}_k(\mathfrak{g} \subset \bar{\mathfrak{g}})$ the distribution of k -planes V^k on $\bar{\mathfrak{g}}$ such that $(\omega_{\bar{\mathfrak{g}}})|_{V^k}$ is equivalent to $\omega_{\mathfrak{g}}$. Let $\hat{\mathcal{D}}_k(\mathfrak{g} \subset \bar{\mathfrak{g}})$ be the distribution on \bar{G} obtained from $\mathcal{D}_k(\mathfrak{g} \subset \bar{\mathfrak{g}})$ by translations composed from left and right multiplications on \bar{G} . Let M^k be an integral submanifold of $\hat{\mathcal{D}}_k(\mathfrak{g} \subset \bar{\mathfrak{g}})$. Then M^k admits

a closed 3-form of type $\omega_{\mathfrak{g}}$, which is the restriction of the Cartan 3-form on \bar{G} . It is an interesting question, if $\mathcal{D}_k(\mathfrak{g} \subset \bar{\mathfrak{g}})$ has an integral submanifold not locally isomorphic to the connected Lie subgroup G having Lie algebra \mathfrak{g} in \bar{G} . Note that $\mathcal{D}_k(\mathfrak{g} \subset \bar{\mathfrak{g}})$ contains a subset $\hat{\mathcal{D}}(\mathfrak{g})$ which is obtained from subspace $\mathfrak{g} \subset \bar{\mathfrak{g}}$ by translations composed from left and right multiplications on \bar{G} . Liu proved that any integral submanifold of $\hat{\mathcal{D}}(\mathfrak{g})$ is a totally geodesic submanifold in \bar{G} [22]. His result generalized a previous result by Ohnita and Tasaki [28].

4. ALGEBRAIC TYPES OF MANIFOLDS PROVIDED WITH A 3-FORM ω^3 OF TYPE $\omega_{\mathfrak{g}}$

In this section we recall the notion of the intrinsic torsion of a G -structure on a manifold M^n , and the notion of the algebraic type of a G -structure, specializing for the case $G = \text{Aut}(\mathfrak{g}) \subset \text{SO}(\mathfrak{g})$, where \mathfrak{g} is a compact simple Lie algebra of dimension at least 8, see Definition 4.2, Definition 4.4. We prove some structure theorems on \mathfrak{g} -submodules of the \mathfrak{g} -module $\mathfrak{g} \otimes \mathfrak{g}^\perp$ with focus on those intrinsic torsions whose affine torsion is skew-symmetric, see Remark 4.5, Remark 4.9, Proposition 4.6, Theorem 4.8 and Corollary 4.12. We prove that any complete torsion-free $\text{Aut}(\mathfrak{g})$ -manifold M^n is either a quotient of \mathbb{R}^n or irreducible and locally symmetric of type I or IV, see Theorem 4.13.

Suppose that M^n is a manifold equipped with a differential 3-form ω^3 of type $\omega_{\mathfrak{g}}$. By Theorem 2.2 M^n is equipped with an $\text{Aut}(\mathfrak{g})$ -structure, and hence with a Riemannian metric $K_{\mathfrak{g}}$ which is associated with the negative of the Killing metric K on \mathfrak{g} , see Remark 4.3.1 below. Let $\hat{\nabla}$ be an $\text{Aut}(\mathfrak{g})$ -connection of the $\text{Aut}(\mathfrak{g})$ -structure on M^n , and ∇^{LV} the Levi-Civita connection of the Riemannian metric $K_{\mathfrak{g}}$. Then $\eta := \hat{\nabla} - \nabla^{LV}$ is a tensor taking value in $T^*M^n \otimes \mathfrak{so}(TM^n, K_{\mathfrak{g}})$, which is called *the torsion tensor of $\hat{\nabla}$* . Using the isomorphism $\mathfrak{so}(\mathfrak{g}) = \Lambda^2(\mathfrak{g})$ we get the following $\text{Ad}(\mathfrak{g})$ -equivariant decomposition

$$(4.1) \quad \mathfrak{so}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^\perp,$$

where

$$(4.2) \quad \mathfrak{g}^\perp := \ker \delta_{\mathfrak{g}} : \Lambda^2(\mathfrak{g}) \rightarrow \mathfrak{g}, v \wedge w \mapsto [v, w].$$

Remark 4.1. Let us define a linear operator

$$(4.3) \quad d_{\mathfrak{g}} : \Lambda^1(\mathfrak{g}) \rightarrow \Lambda^2(\mathfrak{g}), \langle d_{\mathfrak{g}}(v), x \wedge y \rangle := \langle v, [x, y] \rangle,$$

Clearly, $\delta_{\mathfrak{g}}$ is the adjoint of $d_{\mathfrak{g}}$ with respect to the Killing metric $\langle \cdot, \cdot \rangle$.

Denote by $\eta^{\mathfrak{g}}$ the component of the torsion tensor η in $\mathfrak{g} \otimes \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{so}(\mathfrak{g})$.

Definition 4.2. [33, §1] *The intrinsic torsion of an $\text{Aut}(\mathfrak{g})$ -structure on M^n is defined by*

$$(4.4) \quad \xi := \eta - \eta^{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}^\perp.$$

Since any $\text{Aut}(\mathfrak{g})$ -connection on (M^n, ω^3) is obtained from $\hat{\nabla}$ by adding a tensor taking value in $\mathfrak{g} \otimes \mathfrak{g}$, the intrinsic torsion is defined uniquely on $(M^n, \omega_{\mathfrak{g}})$.

Remark 4.3. 1. Given a differential 3-form ω^3 of type $\omega_{\mathfrak{g}}$ on M^n , we define the associated metric $K_{\mathfrak{g}}$ by specifying a linear isomorphism $I_x : T_x M^n \rightarrow \mathfrak{g}$ sending $\omega_{\mathfrak{g}}$ to ω_x^3 . Then $I_x^*(-K)$ is the required Riemannian metric $K_{\mathfrak{g}}(x)$. By theorem 2.2 the obtained metric does not depend on the choice of I_x .

2. (cf [33, Lemma 1.2]) The above definition of algebraic types of $\text{Aut}(\mathfrak{g})$ -structures is a specialization of a definition of algebraic types of G -structures on manifolds M^n , where $G \subset SO(n)$. This scheme has been suggested first by Gray and Hervella for almost Hermitian manifolds [8]. In fact they considered the case of a group G being the stabilizer of a tensor $T \in V$, where V is a tensor space over \mathbb{R}^n with induced action of G , and they looked at the G -type of the tensor ∇T . Since $\hat{\nabla}T = 0$, where as before $\hat{\nabla}$ is a G -connection, we get

$$(4.5) \quad \nabla^{LC}T = \eta(T) = \xi(T) = \sum_i e_i \otimes (\rho_*(\xi(e_i))T) \in \mathfrak{g} \otimes V,$$

where $\rho_* : \mathfrak{so}(n) \rightarrow \mathfrak{so}(V)$ is the differential of the induced embedding $\rho : SO(n) \rightarrow SO(V)$, and (e_i) is an orthonormal basis in \mathfrak{g} , so $\xi(e_i)$ is a contraction of ξ with e_i , which takes value in $\mathfrak{g}^\perp \subset \mathfrak{so}(\mathfrak{g})$. It follows from (4.5) that if $G \subset SO(n)$ is the stabilizer group of a tensor T on \mathbb{R}^n , then the algebraic G -type of ∇T defines the algebraic G -type of the intrinsic torsion ξ of a G -structure and vice versa.

3. In the case T is a 3-form of type $\omega_{\mathfrak{g}} \in \Lambda^3(\mathfrak{g})$ the formula (4.5) has the following simple expression (see the proof of Lemma 4.7.2 below)

$$(4.6) \quad \rho_*(\tau)(\omega_{\mathfrak{g}}) = d_{\mathfrak{g}}\tau,$$

for $\tau \in \mathfrak{so}(\mathfrak{g}) = \Lambda^2(\mathfrak{g})$.

Definition 4.4. Let W be a \mathfrak{g} -submodule in $\mathfrak{g} \otimes \mathfrak{g}^\perp$. A manifold M^n provided with a 3-form ω^3 of type $\omega_{\mathfrak{g}}$ is called of *algebraic type of W* , if its intrinsic torsion ξ takes value in W .

Remark 4.5. 1. For a compact simple Lie algebra \mathfrak{g} we have computed the decomposition of $\mathfrak{g}^\perp \otimes \mathbb{C}$ into irreducible components using table 1 and table 5 in [27]. We put the result in Table 1. We keep notations in [27] with $k\pi_i$ denoting the irreducible representation of the highest weight $(0, \dots, k_{(i)}, \dots, 0)$ with respect to a basis of simple roots of \mathfrak{g} , and $R(\Lambda)$ denotes the irreducible representation with the highest weight Λ .

Table 1: *Decomposition of $\mathfrak{g}^\perp \otimes \mathbb{C}$.* We denote by $\delta(\mathfrak{g})$ the highest weight of the adjoint representation of $\mathfrak{g} \otimes \mathbb{C}$. We note that $\mathfrak{so}(6) \cong \mathfrak{su}(4)$.

\mathfrak{g}	$\dim \mathfrak{g}$	$\delta(\mathfrak{g})$	$\mathfrak{g}^\perp \otimes \mathbb{C}$
$\mathfrak{su}(n+1)$	$n^2 + 2n$	$\pi_1 + \pi_l$	$R(2\pi_1 + \pi_{n-1}) + R(\pi_2 + 2\pi_n)$
$\mathfrak{so}(2n+1)$	$n(2n+1)$	π_2	$R(\pi_3 + \pi_1)$
$\mathfrak{sp}(n)$	$n(2n+1)$	$2\pi_1$	$R(2\pi_1 + \pi_2)$
$\mathfrak{so}(2n), n \geq 4$	$n(2n-1)$	π_2	$R(\pi_3 + \pi_1)$
E_6	78	π_6	$R(\pi_3)$
E_7	133	π_6	$R(\pi_5)$
E_8	248	π_1	$R(\pi_2)$
F_4	52	π_4	$R(\pi_3)$
G_2	14	π_2	$R(3\pi_1)$

Thus except $\mathfrak{g} = \mathfrak{su}(n+1)$, in all other cases $\mathfrak{g}^\perp \otimes \mathbb{C}$ is irreducible. (I thank Dmitri Panyushev, who informed me this observation). In case $\mathfrak{g} = \mathfrak{su}(n+1)$, since $R(2\pi_1 + \pi_{n-1})$ and $R(\pi_2 + 2\pi_n)$ are complex conjugate, the real $\mathfrak{su}(n+1)$ -module \mathfrak{g}^\perp is also irreducible. We refer the reader to [26, §8] for a comprehensive exposition of the theory of real representations of real semisimple Lie algebras.

2. To find a decomposition of the tensor product $(\mathfrak{g} \otimes \mathfrak{g}^\perp) \otimes \mathbb{C}$ into irreducible components for a given simple Lie algebra \mathfrak{g} we could use available software program packages (GAP or LiE or something else). Note that a general formula for a compact simple Lie algebra \mathfrak{g} in any infinite series A_n, B_n, C_n, D_n is not known. Below we will find some important \mathfrak{g} -submodules in the \mathfrak{g} -module $\mathfrak{g} \otimes \mathfrak{g}^\perp$. First we note that the irreducible component with the largest dimension of the tensor product $R(\delta(g)) \otimes R(\Lambda)$ has the highest weight $\delta(\mathfrak{g}) + \Lambda$ [4, Theorem 3.1]. (In [4, Theorem 3.1] Dynkin gave a simple method to find some less obvious irreducible components of the tensor product of two irreducible complex representations.) Let Λ^\perp denote the highest weight of the irreducible representation $\mathfrak{g}^\perp \otimes \mathbb{C}$ if $\mathfrak{g} \neq \mathfrak{su}(n+1)$, and let Λ^\perp denote the weight $(2\pi_1 + \pi_{n-1})$, if $\mathfrak{g} = \mathfrak{su}(n+1)$. It follows that $\mathfrak{g} \otimes \mathfrak{g}^\perp$ contains an irreducible component $R_{\max} := \text{Re}(R(\delta + \Lambda^\perp))$ if $\mathfrak{g} \neq \mathfrak{su}(n+1)$, and $R_{\max} := R(\delta + \Lambda^\perp) \otimes \mathbb{R}$ if $\mathfrak{g} = \mathfrak{su}(n+1)$. Here we denote by $\text{Re}(V)$ a real form of a complex vector space V , and by $V \otimes \mathbb{R}$ - its realification.

Let us consider the following $\text{Ad}(\mathfrak{g})$ -equivariant linear maps

$$(4.7) \quad D_+ : \mathfrak{g} \otimes \mathfrak{g}^\perp \rightarrow \Lambda^4(\mathfrak{g}) : v \otimes \tau \mapsto \sum v \wedge \rho_*(\tau)(\omega_{\mathfrak{g}}),$$

$$(4.8) \quad D_- : \mathfrak{g} \otimes \mathfrak{g}^\perp \rightarrow \Lambda^2(\mathfrak{g}) : v \otimes \tau \mapsto \sum v \lrcorner \rho_*(\tau)(\omega_{\mathfrak{g}}),$$

where $\rho_*(\tau)$ acts on the space $\Lambda^3(\mathfrak{g})$ as we have explained in Remark 4.3.

Proposition 4.6. 1. A 3-form ω^3 of type $\omega_{\mathfrak{g}}$ is closed, if and only if its intrinsic torsion ξ satisfies $\xi(x) \in \ker D_+(x)$ for all $x \in M^n$.

2. A 3-form ω^3 of type $\omega_{\mathfrak{g}}$ is co-closed, if and only if its intrinsic torsion ξ satisfies $\xi(x) \in \ker D_-(x)$ for all $x \in M^n$.

3. The irreducible component R_{\max} belongs to $\ker(D_-)$.

4. The irreducible component R_{\max} belongs to $\ker(D_+)$ if and only if $\mathfrak{g} = \mathfrak{su}(3)$.

Proof. Let us explain the meaning of the operators D_{\pm} . We denote by (e_i) an orthonormal frame in \mathfrak{g} and (e^i) its dual frame. Using the following well-known identities for a differential form ρ on a Riemannian manifold M^n

$$(4.9) \quad d\rho(x) = \sum_i e^i \wedge \nabla_{e_i}^{LV} \rho(x),$$

$$(4.10) \quad d^* \rho(x) = \sum_i e_i \lrcorner \nabla_{e_i}^{LC} \rho(x),$$

taking into account (4.5), we get the first and the second assertions of Proposition 4.6.

Since \mathfrak{g} and \mathfrak{g}^{\perp} are the only irreducible components of $\Lambda^2(\mathfrak{g})$, the image $\text{Im}(D_-) \subset \Lambda^2(\mathfrak{g})$ does not contain any component equivalent to R_{\max} . It follows that $R_{\max} \subset \ker(D_-)$. This proves the third assertion of Proposition 4.6.

4. We will show that $R_{\max} \subset \ker(D_+)$ if and only if $\mathfrak{g} = \mathfrak{su}(3)$. First we recall that the $\text{Ad}(\mathfrak{g})$ -equivariant differential $d_{\mathfrak{g}} : \Lambda^k(\mathfrak{g}) \rightarrow \Lambda^{k+1}(\mathfrak{g})$ is defined as follows. For $k = 1$ we define $d_{\mathfrak{g}}$ as in Remark 4.1. For $k \geq 2$ we define $d_{\mathfrak{g}}$ inductively by

$$(4.11) \quad d_{\mathfrak{g}}(\alpha \wedge \phi) := d_{\mathfrak{g}}(\alpha) \wedge \phi + (-1)^{\deg \alpha} d_{\mathfrak{g}}(\phi).$$

Next we define linear operators $\Pi_{1,3}^+ : \mathfrak{g} \otimes \Lambda^3(\mathfrak{g}) \rightarrow \Lambda^4(\mathfrak{g})$ and $\Pi_{1,3}^- : \mathfrak{g} \otimes \Lambda^3(\mathfrak{g}) \rightarrow \Lambda^2(\mathfrak{g})$ by

$$\Pi_{1,3}^+(v \otimes \theta) := v \wedge \theta, \quad \Pi_{1,3}^-(v \otimes \theta) := v \lrcorner \theta.$$

Lemma 4.7. 1. The operator $d_{\mathfrak{g}}$ can be expressed by

$$(4.12) \quad d_{\mathfrak{g}}(\phi) = \sum_i (e_i \lrcorner \omega_{\mathfrak{g}}) \wedge (e_i \lrcorner \phi),$$

where (e_i) is an orthonormal basis in \mathfrak{g} .

2. Let $\tau \in \mathfrak{g}^{\perp}$. Then

$$(4.13) \quad D_+(v \otimes \tau) = \Pi_{1,3}^+(v \otimes d_{\mathfrak{g}}(\tau)), \quad D_-(v \otimes \tau) = \Pi_{1,3}^-(v \otimes d_{\mathfrak{g}}(\tau)).$$

Proof. 1. We check easily that the validity of (4.12) for $\phi \in \Lambda^1(\mathfrak{g})$. Denote the RHS of (4.12) by $\sigma_+(\phi)$. We observe that σ_+ is a differential, i.e. $\sigma_+(\alpha \wedge \beta) = \sigma_+(\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \sigma_+(\beta)$. Hence $d_{\mathfrak{g}} = \sigma_+$. This proves the first assertion of Lemma 4.7.

2. Note that D_+ is the restriction of the linear operator also denoted by $D_+ : \mathfrak{g} \otimes \Lambda^2(\mathfrak{g})$ defined by the same formula in (4.7). Thus it suffices to check the validity of (4.13) for basis elements $e_i \otimes (e_j \wedge e_k) \in \mathfrak{g} \otimes \Lambda^2(\mathfrak{g})$. Using (4.7) we get

$$D_+(e_i \otimes (e_j \wedge e_k)) = e_i \wedge [e_j \wedge (e_k \lrcorner \omega_{\mathfrak{g}}) - e_k \wedge (e_j \lrcorner \omega_{\mathfrak{g}})].$$

Using (4.12) we get

$$\Pi_{1,3}^+(e_i \otimes d_{\mathfrak{g}}(e_j \wedge e_k)) = e_i \wedge [(e_k] \omega_{\mathfrak{g}}) \wedge e_j - e_k \wedge (e_j] \omega_{\mathfrak{g}})].$$

Comparing the above formulas we get

$$D_+(v \otimes \tau) = \Pi_{1,3}^+(v \otimes \tau).$$

In the same way we get $D_-(v \otimes \tau) = \Pi_{1,3}^-(v \otimes \tau)$. This completes the proof of the second assertion of Lemma 4.7. \square

Let us continue the proof of Proposition 4.6. Note that the component $R_{\max} \otimes \mathbb{C}$ has the highest vector $v_{\delta} \otimes (v_{\delta} \wedge v_{\delta-\alpha_1})$, where α_1 is a simple root of $\mathfrak{g} \otimes \mathbb{C}$ such that $(\delta, \alpha_1) \neq 0$, and $v_{\delta} \in \mathfrak{g} \otimes \mathbb{C}$ (resp. to $v_{\delta-\alpha_1} \in \mathfrak{g} \otimes \mathbb{C}$) is the root vector corresponding to δ (resp. $\delta - \alpha_1$), see e.g. [4, Theorem 3.1]. To prove the last assertion of Proposition 4.6 it suffices to show that $D_+(e_{\delta} \otimes (e_{\delta} \wedge e_{\delta-\alpha_1})) = 0$ if and only if $\mathfrak{g} = \mathfrak{su}(3)$, where D_+ , $\Pi_{1,3}^+$ and $d_{\mathfrak{g}}$ extend \mathbb{C} -linearly on $(\mathfrak{g} \otimes \mathfrak{g}^{\perp}) \otimes \mathbb{C}$ and $\Lambda^k(\mathfrak{g} \otimes \mathbb{C})$. A direct computation using (4.13) shows that

$$D_+(e_{\delta} \otimes (e_{\delta} \wedge e_{\delta-\alpha_1})) = e_{\sigma} \wedge \sum_{\alpha+\beta=\sigma} c_{\alpha,\beta}^{\sigma} e_{\alpha} \wedge e_{\beta} \wedge e_{\sigma-\alpha_1},$$

where $d_{\mathfrak{g}}e_{\sigma} = \sum_{\alpha+\beta=\sigma} c_{\alpha,\beta}^{\sigma} e_{\alpha} \wedge e_{\beta}$. Using the table of simple roots of simple Lie algebras and the above formula, we conclude that $D_+(e_{\delta} \otimes (e_{\delta} \wedge e_{\delta-\alpha_1})) = 0$, if and only if $\mathfrak{g} = \mathfrak{su}(3)$. This completes the proof of Proposition 4.6. \square

We introduce new notations by looking at the following orthogonal decompositions

$$W_{har} := \ker(D_+) \cap \ker(D_-) \subset \mathfrak{g} \otimes \mathfrak{g}^{\perp}.$$

$$\ker(D_+) := W_{har} \oplus W_{har}^{\perp}(D_+).$$

$$\ker(D_-) := W_{har} \oplus W_{har}^{\perp}(D_-).$$

Denote by $\delta_{\mathfrak{g}}$ the adjoint of $d_{\mathfrak{g}}$ with respect to the minus Killing metric on \mathfrak{g} . Note that we have the following orthogonal decomposition

$$(4.14) \quad \Lambda^3(\mathfrak{g}) = \langle \omega_{\mathfrak{g}} \rangle_{\mathbb{R}} \oplus \Lambda_d^3(\mathfrak{g}) \oplus \Lambda_{\delta}^3(\mathfrak{g}),$$

where $\Lambda_{\delta}^3(\mathfrak{g}) := \delta_{\mathfrak{g}}(\Lambda^4(\mathfrak{g}))$ and $\Lambda_d^3(\mathfrak{g}) = d_{\mathfrak{g}}(\mathfrak{g}^{\perp})$.

Theorem 4.8. *1. We have the following decomposition*

$$\mathfrak{g} \otimes \mathfrak{g}^{\perp} = W_{har} \oplus_i V_i,$$

where V_i is one of irreducible modules entered in $\Lambda^2(\mathfrak{g}) \oplus \Lambda^4(\mathfrak{g})$.

2. $W_{har}^{\perp}(D_+)$ contains a \mathfrak{g} -module which is isomorphic to $\mathfrak{g}^{\perp} \subset \Lambda^2(\mathfrak{g})$.

3. $W_{har}^{\perp}(D_-)$ contains a \mathfrak{g} -module which is isomorphic to $\Lambda_{\delta}^3(\mathfrak{g})$.

4. The image of D_+ contains the module $\omega_{\mathfrak{g}} \wedge \Lambda^1(\mathfrak{g})$.

5. D_- is surjective.

Proof. 1. We have a decomposition $\mathfrak{g} \otimes \mathfrak{g}^\perp = \ker(D_+) \oplus \ker(D_+)^\perp$. Let us consider the decomposition $\ker(D_+) = W_{har} \oplus W_{har}^\perp(D_+)$. Since the restriction of D_- to $W_{har}^\perp(D_+)$ is injective, $W_{har}^\perp(D_+)$ is isomorphic to a \mathfrak{g} -module in $\Lambda^2(\mathfrak{g})$. Since $(\ker D_+)^\perp$ is isomorphic to a \mathfrak{g} -submodule in $\Lambda^4(\mathfrak{g})$, we get the first assertion of Theorem 4.8 immediately.

2. We define an $Ad(\mathfrak{g})$ -equivariant linear map

$$(4.15) \quad \Theta : \Lambda^3(\mathfrak{g}) \rightarrow \mathfrak{g} \otimes \mathfrak{g}^\perp, T \mapsto \sum_i e_i \otimes \Pi_{\mathfrak{g}^\perp}(e_i \lrcorner T),$$

where $\Pi_{\mathfrak{g}^\perp}$ is the orthogonal projection onto the subspace $\mathfrak{g}^\perp \subset \Lambda^2(\mathfrak{g})$ and (e_i) is an orthonormal basis in \mathfrak{g} .

Remark 4.9. A result by Friedrich in [5, Theorem 3.1] implies that the image $\Theta(\Lambda^3(\mathfrak{g}))$ consists of all intrinsic torsions ξ^A of $Aut(\mathfrak{g})$ -connections A whose affine torsion $T^A := \nabla_X^A Y - \nabla_Y^A X - [X, Y]$ is skew-symmetric, i.e. $\langle T^A(X, Y), Z \rangle = -\langle T^A(X, Z), Y \rangle$.

Lemma 4.10. *For any $\theta \in \Lambda^3(\mathfrak{g})$ we have*

1. $(D_+) \circ \Theta(\theta) = -3d_{\mathfrak{g}}\theta$.
2. $(D_-) \circ \Theta(\theta) = -\delta_{\mathfrak{g}}(\theta)$.

Proof. 1. Let $\theta \in \Lambda^3(\mathfrak{g})$. Using (4.13) we get

$$(4.16) \quad D_+ \circ \Theta(\theta) = \sum_i (e_i \wedge d_{\mathfrak{g}}(e_i \lrcorner \theta)),$$

where (e_i) is an orthonormal basis in \mathfrak{g} . Applying the Cartan formula $d_{\mathfrak{g}}(e_i \lrcorner \phi) = -e_i \lrcorner d_{\mathfrak{g}}\phi + ad_{e_i}\phi$ to the RHS of (4.16) we get

$$(4.17) \quad D_+ \circ \Theta(\theta) = \sum_i e_i \wedge [(-e_i \lrcorner d_{\mathfrak{g}}\theta) + ad_{e_i}(\theta)] = -4d_{\mathfrak{g}}\theta + \sum_i e_i \wedge ad_{e_i}(\theta).$$

Set $\tilde{d}_{\mathfrak{g}} := \sum_i e_i \wedge ad_{e_i}$. Note that the operator $\tilde{d}_{\mathfrak{g}} : \Lambda^k(\mathfrak{g}) \rightarrow \Lambda^{k+1}(\mathfrak{g})$ is a differential, i.e. $\tilde{d}_{\mathfrak{g}}(\alpha \wedge \beta) = \tilde{d}_{\mathfrak{g}}(\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \tilde{d}_{\mathfrak{g}}\beta$. Furthermore we check easily for $\alpha \in \Lambda^1(\mathfrak{g})$ that $\tilde{d}_{\mathfrak{g}}(\alpha) = d_{\mathfrak{g}}(\alpha)$. It follows that $d_{\mathfrak{g}} = \tilde{d}_{\mathfrak{g}}$. Using this we get the first assertion of Lemma 4.10 immediately from (4.17).

2. Let $\theta \in \Lambda^3(\mathfrak{g})$. Using (4.13) and the Cartan formula $ad_{e_i}\theta = d_{\mathfrak{g}}(e_i \lrcorner \theta) + e_i \lrcorner d_{\mathfrak{g}}\theta$, we get

$$(4.18) \quad D_- \circ \Theta(\theta) = \sum_i (e_i \lrcorner d_{\mathfrak{g}}(e_i \lrcorner \theta)) = \sum_i (e_i \lrcorner ad_{e_i}\theta).$$

On the other hand we have

$$(4.19) \quad \delta_{\mathfrak{g}}(v_1 \wedge \cdots \wedge v_k) = \sum_{i < j} (-1)^{i+j+1} [v_i, v_j] \wedge v_1 \wedge \cdots \hat{v}_i \cdots \hat{v}_j \cdots \wedge v_k.$$

Now compare (4.18) with (4.19) we get the second assertion of Lemma 4.10 immediately. \square

Lemma 4.11. $\ker \Theta = \langle \omega_{\mathfrak{g}} \rangle_{\mathbb{R}}$.

Proof. Clearly $\phi \in \ker \Theta$ if and only if $v_i \lrcorner \phi = d_{\mathfrak{g}} w_i$ for all v_i and some $w_i \in \mathfrak{g}$ depending on v_i . In particular $\omega_{\mathfrak{g}} \in \ker \Theta$.

Now let $\phi \in \ker \Theta$. We write

$$\phi = \phi_{\text{harm}} + \phi_d + \phi_{\delta}$$

corresponding to the decomposition (4.14). To complete the proof of Lemma 4.11 it suffices to show that $\phi_d = 0 = \phi_{\delta}$. Using Lemma 4.10 we get

$$(D_+) \circ \Theta(\phi) = -3d_{\mathfrak{g}} \phi_{\delta},$$

$$(D_-) \circ \Theta(\phi) = -\delta_{\mathfrak{g}} \phi_d.$$

Hence we get

$$d_{\mathfrak{g}} \phi_{\delta} = 0 = \delta_{\mathfrak{g}} \phi_d.$$

Since $(\ker d_{\mathfrak{g}})|_{\Lambda_d^3(\mathfrak{g})} = 0$ and $(\ker \delta_{\mathfrak{g}})|_{\Lambda_{\delta}^3(\mathfrak{g})} = 0$, we conclude that $\phi_d = 0 = \phi_{\delta}$. This completes the proof of Lemma 4.11. \square

Let us continue the proof of Theorem 4.8.2. It follows from Lemma 4.10

$$(4.20) \quad D_+(\Theta(d_{\mathfrak{g}}(\mathfrak{g}^{\perp}))) = 0,$$

$$(4.21) \quad D_-(\Theta(d_{\mathfrak{g}}(\tau))) = -\delta_{\mathfrak{g}} d_{\mathfrak{g}}(\tau) \text{ for } \tau \in \mathfrak{g}^{\perp}.$$

It follows that $\Theta(d_{\mathfrak{g}}(\mathfrak{g}^{\perp})) \subset W_{\text{har}}^{\perp}(D_+)$. Taking into account Lemma 4.11 this proves the second assertion of Theorem 4.8 immediately.

3. By Lemma 4.11 $\ker \Theta|_{\Lambda_{\delta}^3(\mathfrak{g})} = 0$. Lemma 4.10 implies that $\ker(D_-)$ contains a \mathfrak{g} -submodule $\Theta(\Lambda_{\delta}^3(\mathfrak{g}))$, and moreover the kernel of the restriction of D_+ to $\Theta(\Lambda_{\delta}^3(\mathfrak{g}))$ is zero. This proves the third assertion of Theorem 4.8.

4. The fourth assertion of Theorem 4.8 follows by comparing (4.7) with the following formula

$$\sum_i e_i \wedge \rho_*(e_i \wedge X)(\theta) = 3X \wedge \theta,$$

where (e_i) is an orthonormal basis in \mathfrak{g} , $X \in \mathfrak{g}$ and $\theta \in \Lambda^3(\mathfrak{g})$, in particular this formula holds for $\theta = \omega_{\mathfrak{g}}$.

5. Using (4.21) we conclude that the image of D_- contains \mathfrak{g}^{\perp} . A direct computation yields the following identity for any $X \in \mathfrak{g}$

$$\sum_i e_i \lrcorner \rho_*(e_i \wedge X)(\omega_{\mathfrak{g}}) = -2X \lrcorner \omega_{\mathfrak{g}} = -2d_{\mathfrak{g}} X,$$

where (e_i) is an orthonormal basis in \mathfrak{g} . It follows that the image of D_- contains the irreducible component $d_{\mathfrak{g}} \mathfrak{g} \subset \Lambda^2(\mathfrak{g})$. Since $\Lambda^2(\mathfrak{g}) = d_{\mathfrak{g}}(\mathfrak{g}) \oplus \mathfrak{g}^{\perp}$, this completes the proof of Theorem 4.8. \square

From Remark 4.9, Lemma 4.11 and Lemma 4.10 we get immediately

Corollary 4.12. *The space of $Aut(\mathfrak{g})$ -connections with skew-symmetric affine torsion on a manifold M^n provided with a 3-form ω^3 of type $\omega_{\mathfrak{g}}$ is a direct sum of two \mathfrak{g} -modules, one of them consists of those connections for which $d\omega^3 = 0$ and the other one consists of those connections for which $d^*\omega^3 = 0$.*

It follows that a manifold M^n admitting a harmonic form ω^3 of type $\omega_{\mathfrak{g}}$ and having an $Aut(\mathfrak{g})$ -connection with skew-symmetric torsion is in fact torsion-free.

Theorem 4.13. *Let M^n be a complete torsion-free $Aut(\mathfrak{g})$ -manifold. Then M^n is either flat, or M^n is irreducible and locally symmetric of type I or IV.*

Proof. Let \mathfrak{h} be a Lie subalgebra in \mathfrak{g} . We write $\mathfrak{g} = \mathfrak{h} \oplus V$, where V is orthogonal complement to \mathfrak{h} . Since \mathfrak{g} is simple, the adjoint representation $ad_{\mathfrak{g}}(\mathfrak{h})$ restricted to V is nontrivial. Taking into account the de Rham decomposition theorem, we conclude that M^n cannot have a holonomy group H strictly smaller than $Ad(\mathfrak{g})$ unless $H = Id$. Hence M^n must be either irreducible and locally symmetric, or flat. Taking into account Theorem 8.1 we conclude that, if M^n is locally symmetric, then it is of type I or IV. This completes the proof of Theorem 4.13. \square

Remark 4.14. 1. A complete description of algebraic types of $PSU(3)$ -structures is given in Witt's Ph.D. Thesis [35]. In [29] Puhle studies the algebraic types of $PSU(3)$ -structures in greater detail.

2. We could extend many results in this section to the case of simple non-compact Lie group, using the theory of real representation of semisimple Lie algebras [26].

5. COHOMOLOGY THEORIES FOR $Aut^+(\mathfrak{g})$ -MANIFOLDS EQUIPPED WITH A QUASI-CLOSED FORM OF TYPE ϕ_0^l

In this section we also assume that \mathfrak{g} is a compact simple Lie algebra of dimension $n \geq 8$. Let $Aut^+(\mathfrak{g}) := Aut(\mathfrak{g}) \cap GL^+(\mathfrak{g})$ and ϕ_0^l an $Aut^+(\mathfrak{g})$ -invariant l -form on \mathfrak{g} . We study necessary and sufficient conditions for an orientable $Aut(\mathfrak{g})$ -manifold to admit a multi-symplectic form ϕ^l of type ϕ_0^l satisfying $d\phi^l = \theta \wedge \phi^l$, see Theorem 5.1 and Lemma 5.3. Such a differential form ϕ^l will be called a *quasi-closed form*. Furthermore, we construct cohomology groups for two differential complexes $(\Omega_{\phi_{\pm}^l}^*(M^n), d_{\pm})$ on an $Aut^+(\mathfrak{g})$ -manifold M^n provided with a quasi-closed form ϕ^l of type ϕ_0^l , see Proposition 5.8. We consider a spectral sequence relating these cohomologies with the deRham cohomologies of M^n , see Theorem 5.9. We compute these groups in the case of 8-manifolds admitting a harmonic 3-form of type $\omega_{su(3)}$ in Example 5.11. We introduce the notion of ϕ_-^l -harmonic forms, see Definition 5.13, and show some relations between ϕ_-^l -harmonic forms and the group $H_{\phi_-^l}^*(M^n)$, see Proposition 5.14 and Lemma 5.17.

Theorem 5.1. 1. Any $Ad(\mathfrak{g})$ -invariant form ϕ_0^l on \mathfrak{g} is multi-symplectic, if \mathfrak{g} is a simple Lie algebra over \mathbb{C} or over \mathbb{R} .

2. Let \mathfrak{g} be a classical compact simple Lie algebra. Then the algebra $\Lambda_{Aut^+}(\mathfrak{g})$ of $Aut^+(\mathfrak{g})$ -invariant forms on \mathfrak{g} coincides with the algebra $\Lambda_{\mathfrak{g}}(\mathfrak{g})$ of $Ad(\mathfrak{g})$ -invariant forms on \mathfrak{g} except the case $\mathfrak{g} = \mathfrak{su}(n+1)$ where 4 divides $n(n+3)$.

3. Let $\mathfrak{g} = \mathfrak{su}(n+1)$ such that 4 divides $n(n+3)$. Then $\Lambda_{Aut^+}(\mathfrak{g}) = \Lambda(x_{4k-1}, x_{4l+1}, x_{4m+1})$ where $x_{4p\pm 1}$ are primitive generators of $\Lambda_{\mathfrak{g}}(\mathfrak{g})$.

Proof. Recall that a form $\phi_0^l \in \Lambda^l(\mathfrak{g}^*) \cong \Lambda^l(\mathfrak{g})$ is multi-symplectic if and only the map

$$L_{\phi_0^l} : \mathfrak{g} \rightarrow \Lambda^{l-1}(\mathfrak{g}) : v \mapsto v \lrcorner \phi_0^l$$

is injective. Since ϕ_0^l is $Ad(\mathfrak{g})$ -invariant, the kernel of $L_{\phi_0^l}$ can be either 0 or \mathfrak{g} . Since $\phi_0^l \neq 0$ there exists $v \in \mathfrak{g}$ such that $v \lrcorner \phi_0^l \neq 0$. Hence $\ker L_{\phi_0^l} = 0$, this proves the first assertion of Theorem 5.1.

Let $\mathfrak{g} = \mathfrak{su}(n+1)$. It is known that $Aut(\mathfrak{g})$ is generated by $Ad(\mathfrak{g})$ and the complex conjugation σ on $\mathfrak{su}(n+1)$. We can check easily that σ is orientation preserving if and only if 4 divides $n(n+3)$, see also Remark 5.2 below. Clearly σ acts on the primitive elements x_{2i+1} by multiplying x_{2i+1} with ± 1 . To find exactly the sign of this multiplication we note that

$$(\sigma(\phi_0^l), v) = (\phi_0^l, \sigma(v))$$

for any l -vector $v \in \Lambda^l(\mathfrak{su}(n+1))$. Take the unit $(2i+1)$ -vector associated with the orthogonal complement to $\mathfrak{su}(i)$ in $\mathfrak{su}(i+1) \subset \mathfrak{su}(n+1)$ as v , we conclude that

$$\begin{aligned} \sigma(x_{2i+1}) &= -x_{2i+1}, \text{ if } i = 2l, \\ \sigma(x_{2i+1}) &= x_{2i+1} \text{ if } i = 2l - 1. \end{aligned}$$

This proves the third assertion and the part of the second assertion of Theorem 5.1 concerning $\mathfrak{g} = \mathfrak{su}(n+1)$.

Let $\mathfrak{g} = \mathfrak{so}(2n)$. It is known that $Aut(\mathfrak{g})$ is generated by $Ad(\mathfrak{g})$ and the element $\sigma = Ad(diag(1, \dots, 1, -1)) \in Ad(O(2n))$. Clearly σ reserves the orientation. Hence $Aut^+(\mathfrak{g}) = Ad(\mathfrak{g})$, if $\mathfrak{g} = \mathfrak{so}(2n)$ and $n \neq 4$. Combining with Remark 2.4.2 we obtain easily the second assertion of Theorem 5.1 for the case $\mathfrak{g} \neq \mathfrak{so}(8)$.

Note that the group $Aut(\mathfrak{so}(8))$ is generated by $Ad(\mathfrak{so}(8))$ and Σ_3 , see Remark 2.4.2. Since Σ_3 is generated by σ_i , $i = \overline{1, 3}$, which is conjugate by an element in $Aut(\mathfrak{so}(8))$ to the element $\sigma = Ad(diag(1, \dots, 1, -1)) \in Ad(O(8))$, it follows that $Aut^+(\mathfrak{so}(8))$ acts on $\Lambda_{\mathfrak{so}(8)}(\mathfrak{so}(8))$ as identity. This completes the proof of Theorem 5.1. \square

Remark 5.2. It has been observed by Todor Milev that an outer automorphism σ of a simple compact Lie algebra \mathfrak{g} is orientation preserving, if and only if the action of σ on the Dynkin diagram $D(\mathfrak{g})$ is a composition of even number of permutations. To prove this statement for classical compact Lie algebras he used an argument similar to our argument above. For

the case E_6 he proved this assertion with a help of a computer program written by himself. In particular $Aut^+(E_6) = Aut(E_6)$. We conjecture that $\Lambda_{Aut^+}(E_6) = \Lambda(x_3, x_{11}, x_{15}, x_9x_{17}, x_{23})$.

Now assume that $\phi_0^l \in \Lambda_{Aut^+}(\mathfrak{g})$. Recall that M^n admits a differential form ϕ^l of type ϕ_0^l , which by Theorem 5.1 is multi-symplectic. Let ξ be the intrinsic torsion of the $Aut^+(\mathfrak{g})$ -structure on M^n . As in the previous sections we denote by ω^3 the Cartan 3-form on M^n .

Lemma 5.3. *1. The values of $d\phi^l$ and $d^*\phi^l$ depend linearly on the intrinsic torsion ξ of M^n .*

$$(5.1) \quad d\phi^l(x) = \sum_i e_i \wedge \rho_*(\xi(e_i))(\phi^l(x)),$$

$$(5.2) \quad d^*\phi^l(x) = \sum_i e_i \lrcorner \rho_*(\xi(e_i))(\phi^l(x)),$$

where (e_i) is an orthonormal basis in $T_x M^n$, see also (4.7), (4.8).

2. If $d\phi^l = \phi^l \wedge \theta$ then θ is defined uniquely by the following formula

$$(5.3) \quad \theta = c(\phi_0^l) * (\phi \wedge *d\phi),$$

for some nonzero constant $c(\phi_0^l)$.

3. If ω^3 is quasi-closed, i.e. $d\omega^3 = \omega^3 \wedge \theta$, then ω^3 is locally conformally closed: $d\theta = 0$.

Proof. The first assertion of Lemma 5.3 is a direct consequence of (4.5), (4.9) and (4.10). In particular, the existence of a closed form or a quasi closed form of type ϕ_0^l on M^n is defined entirely by the algebraic type of the intrinsic torsion ξ of M^n .

Next we observe that the linear map $L_{\phi^l} : \mathfrak{g} \rightarrow \Lambda^{l+1}(\mathfrak{g})$, $\theta \mapsto \phi^l \wedge \theta$ is an $Ad(\mathfrak{g})$ -equivariant map between two $Ad(\mathfrak{g})$ -irreducible modules, moreover L_{ϕ^l} extends linearly to the complexified irreducible modules of \mathfrak{g} , since \mathfrak{g}^C is simple. Applying the Schur Lemma we obtain the second assertion of Lemma 5.3.

Note that $d^2\omega^3 = 0 = \omega^3 \wedge \theta \wedge \theta - \omega^3 \wedge d\theta = -\omega^3 \wedge d\theta$. By Corollary 5.6.2 proved below $d\theta = 0$. This completes the proof of Lemma 5.3. \square

In what follows we single out several interesting \mathfrak{g} -modules associated with an $Ad(\mathfrak{g})$ -invariant form ϕ_0^l in the exterior algebra $\Lambda(\mathfrak{g})$. If ϕ_0^l is also $Aut^+(\mathfrak{g})$ -invariant, then these modules generate associate $Aut^+(\mathfrak{g})$ -invariant sub-bundles in M^n .

Let ϕ^l be an $Ad(\mathfrak{g})$ -invariant l -form on \mathfrak{g} of degree $3 \leq l \leq n-3$ (for simplicity we drop a lower index 0 at ϕ_0^l when we are dealing exclusively with l -forms on \mathfrak{g}). For each $0 \leq k \leq n$ we define the following linear operator

$$L_{\phi^l} : \Lambda^k(\mathfrak{g}) \rightarrow \Lambda^{k+l}(\mathfrak{g}), \gamma \mapsto \phi^l \wedge \gamma.$$

Now we look at a decomposition of $\Lambda^k(\mathfrak{g})$ under the action of L_{ϕ^l} for $0 \leq k \leq n$. Set

$$(5.4) \quad \Lambda_{\phi_+^l}^k := \{\beta \in \Lambda^k(\mathfrak{g}) \mid L_{\phi^l}(\beta) = 0\},$$

$$(5.5) \quad \Lambda_{\phi_-^l}^k := \{\gamma \in \Lambda^k(\mathfrak{g}) \mid \langle \gamma, \beta \rangle = 0 \forall \beta \in \Lambda_+^k\},$$

here \langle, \rangle denotes the induced inner product on $\Lambda^k(\mathfrak{g})$. Denote by $*$ -the Hodge operator on \mathfrak{g} associated to the Killing metric and some preferred orientation on \mathfrak{g} .

Proposition 5.4. *Assume that $0 \leq k \leq n$. Then*

$$(5.6) \quad \Lambda_{\phi_-^l}^k = *L_{\phi^l}(\Lambda^{n-l-k}(\mathfrak{g})) = *L_{\phi^l}(\Lambda_{\phi_-^l}^{n-l-k}).$$

*The space $\Lambda_{\phi_-^l}^k$ is not zero if and only if $(n-l) \geq k$. The operator $*L_{\phi^l}$ induces an isomorphism between $\Lambda_{\phi_-^l}^k$ and $\Lambda_{\phi_-^l}^{n-l-k}$. In particular we have*

$$(5.7) \quad \Lambda_{\phi_-^l}^0 \cong \Lambda_{\phi_-^l}^l = \mathbb{R}.$$

$$(5.8) \quad \Lambda_{\phi_-^l}^1 \cong \Lambda_{\phi_-^{n-l}}^{l-1} \cong \Lambda^1(\mathfrak{g}).$$

$$(5.9) \quad \Lambda_{\phi_-^l}^k = 0 \text{ for } n-l+1 \leq k \leq n.$$

Furthermore we have the following identities

$$(5.10) \quad (-1)^{l+1}L_{\phi^l}d_{\mathfrak{g}} + d_{\mathfrak{g}}L_{\phi^l} = 0 = (-1)^{l+1}L_{\phi^l}\delta_{\mathfrak{g}} + \delta_{\mathfrak{g}}L_{\phi^l}.$$

Hence operators L_{ϕ^l} preserves the subspaces $\Lambda_{\text{harm}}(\mathfrak{g}), \Lambda_d(\mathfrak{g}), \Lambda_{\delta}(\mathfrak{g})$.

Proof. First we show that $*L_{\phi^l}(\Lambda^{n-l-k}(\mathfrak{g})) \subset \Lambda_{\phi_+^l}^k$. Let $\beta \in \Lambda_{\phi_+^l}^k$. Then

$$\langle \beta, *(\phi^l \wedge \gamma) \rangle = \langle \text{vol}, \beta \wedge \phi^l \wedge \gamma \rangle = 0$$

since $\beta \wedge \phi^l = 0$. Hence $*(\phi^l \wedge \gamma) \in \Lambda_{\phi_-^l}^k$.

Now we show that $\Lambda_{\phi_-^l}^k \subset *L_{\phi^l}(\Lambda^{n-l-k}(\mathfrak{g}))$. It suffices to show that the orthogonal complement of $*L_{\phi^l}(\Lambda^{n-l-k}(\mathfrak{g}))$ in $\Lambda^k(\mathfrak{g})$ is a subset of $\Lambda_{\phi_+^l}^k$. Clearly

$$\beta \in (*L_{\phi^l}(\Lambda^{n-l-k}(\mathfrak{g})))^{\perp} \iff \langle \beta, *(\phi^l \wedge \gamma) \rangle = 0 \text{ for all } \gamma \in \Lambda^{n-l-k}(\mathfrak{g})$$

Then

$$\beta \wedge \phi^l \wedge \gamma = 0 \text{ for all } \gamma \in \Lambda^{n-l-k}(\mathfrak{g}).$$

Hence

$$\beta \wedge \phi^l = 0,$$

which by definition (5.4) implies that $\beta \in \Lambda_{\phi_+^l}^k$. This proves the first assertion (5.6) of Proposition 5.4.

The second assertion follows from the definition (5.5) of $\Lambda_{\phi_-}^k$, taking into account the existence of a nonzero element $\gamma \in \Lambda^{n-l-k}(\mathfrak{g})$ such that $\phi^l \wedge \gamma \neq 0$, if $n - l - k \geq 0$.

The third assertion follows from (5.6), (5.4), (5.5). Clearly the next assertions (5.7) and (5.9) are direct consequences of the second and the third assertion. It remains to prove (5.8). Note that $\Lambda_{\phi_-}^1$ is nonempty by the second assertion. Since $\Lambda^1(\mathfrak{g})$ is an irreducible \mathfrak{g} -module, we get (5.8) from the third assertion of Proposition 5.4.

Next, using $d_{\mathfrak{g}}\phi^l = 0$ we get the first identity in (5.10). To prove the second identity in (5.10) we use the following

Lemma 5.5. [16] *The Schouten-Nijenhuis bracket $\{\cdot, \cdot\}_{\mathfrak{g}}$ on $\Lambda(\mathfrak{g})$ can be expressed in terms of $\delta_{\mathfrak{g}}$ as follows*

$$\delta_{\mathfrak{g}}(A_l \wedge B_m) = \delta_{\mathfrak{g}}(A_l) \wedge B_m + (-1)^l A_l \wedge \delta_{\mathfrak{g}}(B_m) + (-1)^{l+1} \{A_l, B_m\}_{\mathfrak{g}}.$$

Substituting $A_l = \phi^l$ in the above formula, and taking into account $\{\phi^l, B_m\}_{\mathfrak{g}} = 0$ for all $B_m \in \Lambda(\mathfrak{g})$, since ϕ^l is $Ad(\mathfrak{g})$ -invariant, we get

$$(5.11) \quad \delta_{\mathfrak{g}}(\phi^l \wedge B_m) = (-1)^l \phi^l \wedge \delta_{\mathfrak{g}}(B_m),$$

which is equivalent to the second identity in (5.10). This completes the proof of Proposition 5.4. \square

Corollary 5.6. *The following relations hold for any compact simple Lie algebra \mathfrak{g} .*

$$(5.12) \quad \Lambda_{(\omega_{\mathfrak{g}})_-}^2 \cong \Lambda_{(\omega_{\mathfrak{g}})_-}^{n-5} \cong \Lambda^2(\mathfrak{g}) \text{ as } \mathfrak{g}\text{-module.}$$

$$(5.13) \quad \Lambda_{(*\omega_{\mathfrak{g}})_-}^3 \cong \Lambda_{(*\omega_{\mathfrak{g}})_-}^0 \cong \mathbb{R} \text{ as } \mathfrak{g}\text{-module.}$$

$$(5.14) \quad \Lambda_{(\omega_{\mathfrak{g}})_-}^3 \supset d_{\mathfrak{g}}(\mathfrak{g}^{\perp}).$$

$$(5.15) \quad \Lambda_{(\omega_{\mathfrak{g}})_+}^k \supset (\Lambda^{k-3}(\mathfrak{g}) \wedge \omega_{\mathfrak{g}}).$$

Proof. Let us prove (5.12). By Remark 4.5 $\Lambda^2(\mathfrak{g})$ is a sum of two irreducible \mathfrak{g} -sub-modules $d_{\mathfrak{g}}(\Lambda^1(\mathfrak{g}))$ and \mathfrak{g}^{\perp} . Thus it suffices to show that the action of $L_{\omega_{\mathfrak{g}}}$ restricted to each sub-module $d_{\mathfrak{g}}(\Lambda^1(\mathfrak{g}))$ and \mathfrak{g}^{\perp} is not zero.

First we will show that the image $L_{\omega_{\mathfrak{g}}}(d_{\mathfrak{g}}(\Lambda^1(\mathfrak{g}))) \neq 0$. Equivalently it suffices to show that $d_{\mathfrak{g}}(L_{\omega_{\mathfrak{g}}}\Lambda^1(\mathfrak{g})) \neq 0$. By (5.10) we have $\delta_{\mathfrak{g}}L_{\omega_{\mathfrak{g}}}\Lambda^1(\mathfrak{g}) = 0$. Since $b_4(\mathfrak{g}) = 0$ it follows that $L_{\omega_{\mathfrak{g}}}\Lambda^1(\mathfrak{g}) \not\subset \ker d_{\mathfrak{g}}$. Hence $L_{\omega_{\mathfrak{g}}}(d_{\mathfrak{g}}(\Lambda^1(\mathfrak{g}))) \neq 0$.

Next we will show that $L_{\omega_{\mathfrak{g}}}(\mathfrak{g}^{\perp}) \neq 0$. Let Δ be the root system of $\mathfrak{g}^{\mathbb{C}}$. We recall the following root decomposition of the complexification $\mathfrak{g}^{\mathbb{C}}$, where \mathfrak{g} is a compact real form of $\mathfrak{g}^{\mathbb{C}}$, see e.g. [10, Theorem 4.2] and [10, Theorem 6.3]. Let $\mathfrak{h}_0 \in \mathfrak{g}$ be a Cartan subalgebra of \mathfrak{g} , so $\mathfrak{h}_0^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let Δ^+ be a positive root system of $\mathfrak{g}^{\mathbb{C}}$ and $\Sigma \subset \Delta^+$ be a system of simple roots. Denote by $E_{\pm\alpha}$, $\alpha \in \Delta^+$, the corresponding root vectors such

that $[E_\alpha, E_{-\alpha}] = \frac{2H_\alpha}{\alpha(H_\alpha)} \in \mathfrak{h}_0^\mathbb{C}$, see e.g. [10, p.258]. We decompose \mathfrak{g} as

$$(5.16) \quad \mathfrak{g}^\mathbb{C} = \oplus_{\alpha \in \Sigma} \langle H_\alpha \rangle_\mathbb{R} \oplus_{\alpha \in \Delta^+} \langle E_\alpha \rangle_\mathbb{R} \oplus_{\alpha \in \Delta^+} \langle E_{-\alpha} \rangle_\mathbb{R}.$$

Then

$$(5.17) \quad \mathfrak{g} = \oplus_{\alpha \in \Sigma} \langle h_\alpha \rangle_\mathbb{R} \oplus_{\alpha \in \Delta^+} \langle e_\alpha \rangle_\mathbb{R} \oplus_{\alpha \in \Delta^+} \langle f_\alpha \rangle_\mathbb{R}.$$

Now we set $h_\alpha := iH_\alpha$, $e_\alpha := i(E_\alpha + E_{-\alpha})$ and $f_\alpha := (E_\alpha - E_{-\alpha})$.

Note that $L_{\omega_g}(\delta_g(h_\alpha \wedge h_\beta \wedge e_\alpha))$ contains a nonzero summand of form $e_\alpha \wedge e_\beta \wedge e_{\alpha+\beta} \wedge (-c_\alpha h_\beta + c_\beta^\alpha h_\alpha) \wedge f_\alpha$, where $c_\alpha c_\beta^\alpha \neq 0$. Since $\delta_g(\Lambda^3(\mathfrak{g})) = \mathfrak{g}^\perp$ is irreducible, it follows that $\mathfrak{g}^\perp \subset \Lambda_{(\omega_g)_-}^2$. This completes the proof of (5.12).

Clearly (5.13) is a direct consequence of Proposition 5.4.

Now let us prove (5.14). Note that

$$(5.18) \quad \langle \omega_g \rangle_\mathbb{R} \subset \Lambda_{(\omega_g)_+}^3.$$

Next we show that

$$(5.19) \quad L_{\omega_g} d_g(\mathfrak{g}^\perp) \neq 0.$$

By (5.10) we have

$$\delta_g L_{\omega_g}(\mathfrak{g}^\perp) = -L_{\omega_g} \delta_g(\mathfrak{g}^\perp) = 0.$$

It follows that $L_{\omega_g}(\mathfrak{g}^\perp) \subset \ker \delta_g$. Since \mathfrak{g}^\perp is an irreducible module, $L_{\omega_g}(\mathfrak{g}^\perp)$ does not contain any $Ad(\mathfrak{g})$ -invariant form. Hence $L_{\omega_g}(\mathfrak{g}^\perp) \subset \Lambda_\delta^5(\mathfrak{g})$, in particular $d_g(L_{\omega_g}(\mathfrak{g}^\perp)) \neq 0$. This implies (5.19).

The last formula (5.15) is a consequence of the identity $L_{\omega_g}^2 = 0$. This completes the proof of Corollary 5.6. \square

Example 5.7. As a consequence of Proposition 5.4 and Corollary 5.6 we write here the complexes $\Lambda_{(\omega_g)_\pm}$ and $\Lambda_{(*\omega_g)_\pm}$ for $\mathfrak{g} = \mathfrak{su}(3)$.

a	Λ_a^0	Λ_a^1	Λ_a^2	Λ_a^3	Λ_a^4	Λ_a^5	Λ_a^6	Λ_a^7	Λ_a^8
$*\omega_g -$	\mathbb{R}	\mathfrak{g}	$d_g(\mathfrak{g})$	$\langle \omega_g \rangle$	0	0	0	0	0
$*\omega_g +$	0	0	\mathfrak{g}^\perp	$\langle \omega_g \rangle^\perp$	$\Lambda^4(\mathfrak{g})$	$\Lambda^3(\mathfrak{g})$	$\Lambda^2(\mathfrak{g})$	\mathfrak{g}	\mathbb{R}
$\omega_g -$	\mathbb{R}	\mathfrak{g}	$\Lambda^2(\mathfrak{g})$	$d_g(\mathfrak{g}^\perp) \oplus \delta_g(\Lambda_{(\omega_g)_-}^4)$	$*(\Lambda^1(\mathfrak{g}) \wedge \omega_g)$	$\langle *\omega_g \rangle$	0	0	0
$\omega_g +$	0	0	0	$\langle \omega_g \rangle_\mathbb{R} \oplus \mathbb{R}^{27}$	$\mathbb{R}^8 + 2\mathbb{R}^{27}$	$\langle *\omega_g \rangle^\perp$	$\Lambda^2(\mathfrak{g})$	\mathfrak{g}	\mathbb{R}

In this example except the modules $\Lambda_{(\omega_g)_\pm}^3$ all other modules Λ_a^i can be defined easily using Proposition 5.4 and Corollary 5.6. Since $\Lambda^4(\mathfrak{g})$ is invariant under the Hodge star operator $*$ we get the following decomposition

$$(5.20) \quad \Lambda^4(\mathfrak{g}) = (\Lambda^1(\mathfrak{g}) \wedge \omega_g) \oplus \delta_g(\Lambda_{27}^5) + *(\Lambda^1(\mathfrak{g}) \wedge \omega_g) \oplus d_g(\Lambda_{27}^3),$$

where Λ_{27}^3 is an irreducible \mathfrak{g} -submodule of $\Lambda^3(\mathfrak{g}) = \langle \omega_g \rangle_\mathbb{R} \oplus d_g(\mathfrak{g}^\perp) \oplus \Lambda_\delta^3(\mathfrak{g})$, (so $\Lambda_{27}^3 = \Lambda_\delta^3(\mathfrak{g})$) and $\Lambda_{27}^5 = *\Lambda_{27}^3$. Using (5.11) we get $\Lambda^1(\mathfrak{g}) \wedge \omega_g \subset \ker \delta_g$, and hence $*(\Lambda^1(\mathfrak{g}) \wedge \omega_g) \subset \ker d_g$.

Now let us show that

$$(5.21) \quad \delta_{\mathfrak{g}}(\Lambda_{(\omega_{\mathfrak{g}})_-}^4) \subset \Lambda_{(\omega_{\mathfrak{g}})_-}^3.$$

Using (5.11) it suffices to show that $\delta_{\mathfrak{g}}(L_{\omega_{\mathfrak{g}}}(\Lambda_{(\omega_{\mathfrak{g}})_-}^4)) \neq 0$. But that is obvious, since $\ker(\delta_{\mathfrak{g}})|_{\Lambda^7(\mathfrak{g})} = 0$. This yields (5.21).

In the same way we get $L_{\omega}(\delta_{\mathfrak{g}}(\Lambda_{(\omega_{\mathfrak{g}})_+}^4 \cap \ker d_{\mathfrak{g}})) = 0$, which yields

$$(5.22) \quad \delta_{\mathfrak{g}}(\Lambda_{(\omega_{\mathfrak{g}})_+}^4 \cap \ker d_{\mathfrak{g}}) = \Lambda_{27}^3 \subset \Lambda_{(\omega_{\mathfrak{g}})_+}^3.$$

Now the modules $\Lambda_{(\omega_{\mathfrak{g}})_{\pm}}^3$ can be completely defined from Proposition 5.4, Corollary 5.6 and (5.21), (5.22).

Now we assume that ϕ_0^l is an $Aut^+(\mathfrak{g})$ -invariant l -form. Using (5.4) and (5.5) we define the corresponding decomposition of

$$\Omega^k(M) = \Omega_{\phi_+^l}^k(M) \oplus \Omega_{\phi_-^l}^k(M).$$

This leads to the following exact sequence of sheaves

$$(5.23) \quad 0 \rightarrow \Omega_{\phi_+^l}^k(M) \xrightarrow{i} \Omega^k(M) \xrightarrow{p} \Omega_{\phi_-^l}^k(M) \rightarrow 0.$$

Denote by d_- the composition of the differential operator d with the projection Π_- to Π_- to $\Omega_{\phi_-^l}^k(M)$.

Theorem 5.8. *Assume that ϕ^l is quasi-closed. Then*

1. $(\Omega_{\phi_+^l}^*(M), d)$ is a differential sub-complex of $(\Omega^*(M), d)$.
2. $(\Omega_{\phi_+^l}^*(M), d_-)$ is a differential complex.

Proof. 1. First we assume that ϕ^l is quasi-closed. Let us show that $(\Omega_{\phi_+^l}^*(M), d)$ is a differential complex. Assume that $\beta \in \Omega_{\phi_+^l}^k(M)$.

$$(5.24) \quad d\beta \wedge \phi^l = d(\beta \wedge \phi^l) + (-1)^{\deg \beta} \beta \wedge d\phi^l = (-1)^{\deg \beta} \beta \wedge \theta \wedge \phi^l = 0,$$

which implies that

$$(5.25) \quad d\beta \in \Omega_{\phi_+^l}^k(M).$$

This proves the first assertion of Theorem 5.8.

2. Next let us show that $d_-^2 = 0$. Assume that $\alpha \in \Omega_{\phi_-^l}^k(M)$. Since $d\alpha = d_- \alpha + \beta$, where $\beta \in \Omega_{\phi_+^l}^{k+1}(M)$, we get $dd_- \alpha + d\beta = 0$. By (5.25) $d\beta \in \Omega_{\phi_+^l}^{k+2}(M)$, hence $dd_- \alpha \in \Omega_{\phi_+^l}^{k+2}(M)$, which implies that $d_-^2 \alpha = 0$. This completes the proof of Theorem 5.8. \square

Denote by $H^i(M)$ the de Rham cohomology group $H^i(M, \mathbb{R})$, and by $H_{\phi_{\pm}^l}^*(M^n)$ the cohomology groups of sheaves $(\Omega_{\phi_{\pm}^l}^*(M), d_{\pm})$, where d_+ is the restriction of d to $\Omega_{\phi_+^l}(M)$.

Proposition 5.9. *There exists a long exact sequence of cohomology groups*

$$0 \rightarrow H^1(M^n) \xrightarrow{p^*} H^1_{\phi_-^l}(M^n) \xrightarrow{t^*} H^2_{\phi_+^l}(M^n) \xrightarrow{i} H^2(M^n) \xrightarrow{p^*} \dots H^n(M^n) \rightarrow 0.$$

Proof. Let us define the following diagram of chain complexes

$$\begin{array}{ccccc} \Omega_{\phi_+^l}^k(M) & \xrightarrow{d_+} & \Omega_{\phi_+^l}^{k+1}(M) & \xrightarrow{d_+} & \Omega_{\phi_+^l}^{k+2}(M) \xrightarrow{d_+} \\ \downarrow i & & \downarrow i & & \downarrow i \\ \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \xrightarrow{d} & \Omega^{k+2}(M) \xrightarrow{d} \\ \downarrow p & & \downarrow p & & \downarrow p \\ \Omega_{\phi_-^l}^k(M) & \xrightarrow{d_-} & \Omega_{\phi_-^l}^{k+1}(M) & \xrightarrow{d_-} & \Omega_{\phi_-^l}^{k+2}(M) \xrightarrow{d_-} \end{array}$$

To prove Proposition 5.9, taking into account (5.23), it suffices to show that the above diagram is commutative. Equivalently we need to show

$$(5.26) \quad d \circ i = i \circ d_+, \quad d_- \circ p = p \circ d.$$

The first identity in (5.26) is a consequence of (5.25) (the d -closedness of $\Omega_{\phi_+^l}(M)$). The second identity follows from the definition of d_- . This completes the proof of Proposition 5.9. \square

Remark 5.10. The map $p^* : H^k(M) \rightarrow H_{\phi_-^l}^k(M)$ associates each cohomology class of a closed k -form ϕ^k on M to the cohomology class $[p(\phi^k)] \in H_{\phi_-^l}^k(M)$, see (5.26). The connecting homomorphism $t^* : H_{\phi_-^l}^k(M) \rightarrow H_{\phi_+^l}^{k+1}(M)$ is defined as follows: $[\phi] \mapsto [d\phi]$. The map $i^* : H_{\phi_+^l}^k(M) \rightarrow H^k(M)$ associates each cohomology class in $H_{\phi_+^l}^k(M)$ of a closed k -form $\phi^l \in \Omega_{\phi_+^l}^k(M)$ to the cohomology class $[\phi^k] \in H^k(M)$.

Example 5.11. Let us consider the new homology groups arisen in this way on a connected orientable manifold M^8 provided with a 3-form ω^3 of type $\omega_{su(3)}$ assuming that $d\omega^3 = 0 = d^*\omega^3$.

a	$*\omega_{\mathfrak{g}} -$	$*\omega_{\mathfrak{g}} +$	$\omega_{\mathfrak{g}} -$	$\omega_{\mathfrak{g}} +$
H_a^0	\mathbb{R}	0	\mathbb{R}	0
H_a^1	$H^1(M) \oplus \frac{\{d^{-1}(\Omega_{(\omega_{\mathfrak{g}})_+}^2)\}}{\{\ker d \cap \Omega^1(M)\}}$	0	$H^1(M)$	0
H_a^2	$\frac{\Omega_{(*\omega_{\mathfrak{g}})_-}^2}{\Pi_{d_{\mathfrak{g}}}(d\Omega^1(M))}$	$\{\ker d \cap \Omega_{(*\omega_{\mathfrak{g}})_-}^2\}$	$H^2(M) \oplus \frac{\{\phi d\phi \wedge \omega_{\mathfrak{g}}=0\}}{\{\phi d\phi=0\}}$	0
H_a^3	\mathbb{R}	$\frac{H^3(M)}{\mathbb{R}} \oplus d\Omega_{(*\omega_{\mathfrak{g}})_-}^2$	$H_{(\omega_{\mathfrak{g}})_-}^3$	$\{\ker d \cap \Omega_{(\omega_{\mathfrak{g}})_+}^3\}$
H_a^4	0	$H^4(M)$	$\frac{\Omega_{(\omega_{\mathfrak{g}})_-}^4}{d_-(\Omega_{(\omega_{\mathfrak{g}})_-}^3)}$	$H_{(\omega_{\mathfrak{g}})_+}^4(M)$
H_a^5	0	$H^5(M)$	\mathbb{R}	$\frac{H^5(M)}{\mathbb{R}} \oplus d\Omega_{(\omega_{\mathfrak{g}})_-}^4$
H_a^6	0	$H^6(M)$	0	$H^6(M)$
H_a^7	0	$H^7(M)$	0	$H^7(M)$
H_a^8	0	\mathbb{R}	0	\mathbb{R}

In this table we denote by $\Pi_{d_{\mathfrak{g}}}$ the orthogonal projection of $\Omega^*(M)$ on the image of the operator $d_{\mathfrak{g}}$ acting on $\Omega^*(M)$. The cohomology groups $H_{(\omega_{\mathfrak{g}})_-}^3$, $H_{(\omega_{\mathfrak{g}})_+}^4$ are best described by using the relations in Example 5.7.

Remark 5.12. Note that many cohomology groups $H_{\phi_{\omega}}^i$ are infinite-dimensional. Proposition 5.9 implies that the subgroup $p^*(H^i(M)) \subset H_{\phi_-}^i(M^n)$ as well as the coset $H_{\phi_+}^{i+1}/t^*(H_{\phi_-}^i(M))$ are finite-dimensional.

Now we are going to define a subgroup of $H_{\phi_{\pm}}^i(M^n)$.

Definition 5.13. A differential form α in $\Omega_{\phi_{\pm}}^*(M)$ is called ϕ_{\pm} -harmonic, if $d_{\pm}\alpha = 0 = d^*\alpha$.

Denote by $\mathcal{H}^k(M^n)$ the space of all harmonic k -forms on M^n . Let $\mathcal{H}_{\phi_{\pm}}^i(M)$ be the space of all ϕ_{\pm}^l -harmonic forms in $\Omega_{\phi_{\pm}}^k(M^n)$.

Proposition 5.14. *There is a natural monomorphism $h_i^{\pm} : \mathcal{H}_{\phi_{\pm}}^i(M) \rightarrow H_{\phi_{\pm}}^i(M)$ associating each ϕ_{\pm}^l -harmonic form α to its cohomology class $[\alpha] \in H_{\phi_{\pm}}^i(M)$.*

Proof. Assume that $\alpha \in \ker h_i^{\pm}$, i.e. α is a ϕ_{\pm} -harmonic form, and $\alpha = d_{\pm}\gamma$, for some $\gamma \in \Omega_{\phi_{\pm}}^{i-1}(M^n)$. By the definition of d_{\pm} we have

$$\langle \alpha, d_{\pm}\gamma \rangle = \langle \alpha, d\gamma \rangle = \langle d^*\alpha, \gamma \rangle = 0$$

which implies that $\alpha = 0$. This proves Proposition 5.14. \square

By Proposition 5.4, $*L_{\phi^l}$ induces a bundle isomorphism $\Omega_{\phi_-}^1(M^n) \rightarrow \Omega_{\phi_-}^{n-l-1}(M^n)$.

Proposition 5.15. *Assume that $d\phi^l = 0$. The operator $*L_{\phi^l}$ induces an isomorphism also denoted by $*L_{\phi^l} : \mathcal{H}_{\phi_-}^1(M^n) \rightarrow \mathcal{H}_{\phi_-}^{n-l-1}(M^n)$.*

Proof. First we show that if $\alpha \in \mathcal{H}_{\phi_-}^1(M^n)$, then $*L_{\phi^l}(\alpha) \in \mathcal{H}_{\phi_-}^{n-l-1}(M^n)$. We compute

$$(5.27) \quad d^*(*L_{\phi^l}(\alpha)) = d(\alpha \wedge \phi^l) = (d_- \alpha + d_+ \alpha) \wedge \phi^l = 0,$$

since $d_- \alpha = 0$, and $(d_+ \alpha) \wedge \phi^l = 0$.

Next we prove that $d_-(*L_{\phi^l}(\alpha)) = 0$, or equivalently $d(*L_{\phi^l}(\alpha)) \in \Omega_{\phi_+}^{n-l}(M^n)$. Note that we have the following orthogonal Hodge decomposition

$$(5.28) \quad \Omega^{n-l}(M^n) = \mathcal{H}^{n-l}(M^n) \oplus d(\Omega^{n-l-1}(M^n)) \oplus d^*(\Omega^{n-l+1}(M^n)).$$

Since $d\phi^l = 0$, we have $\phi^{n-l} \in \mathcal{H}^{n-l}(M) \oplus d^*(\Omega^{n-l+1}(M^n))$. Hence we get from 5.28

$$(5.29) \quad \langle \phi^{n-l}, d(*L_{\phi^l}(\alpha)) \rangle = d(*L_{\phi^l}(\alpha)) \wedge \phi^l = 0.$$

Hence $d(*L_{\phi^l}(\alpha)) \in \Omega_{\phi_+}^{n-l}(M^n)$. Hence $*L_{\phi^l}(\alpha) \in \mathcal{H}_{\phi_-}^{n-l-1}(M^n)$.

By Proposition 5.4 the restriction of $*L_{\phi^l}$ to $\mathcal{H}_{\phi_-}^1(M^n)$ is also injective.

This proves that $*L_{\phi^l} : \mathcal{H}_{\phi_-}^1(M^n) \rightarrow \mathcal{H}_{\phi_-}^{n-l-1}(M^n)$ is a monomorphism. Let us show that this map is also surjective. We will first prove the following

Lemma 5.16. *There exists a nonzero constant c depending on ϕ_0^l such that for all $\beta \in \Omega_{\phi_-}^{n-l-1}(M^n)$ we have*

$$(5.30) \quad *(\beta \wedge \phi_0^l) \wedge \phi_0^l = c \cdot (*\beta)$$

Proof. Lemma 5.16 follows from the fact that the map

$$\lambda : \Lambda_{(\phi_0^l)_-}^{n-l-1} \rightarrow \Lambda^{n-l-1}(\mathfrak{g}), \quad \beta \mapsto *(\beta \wedge \phi_0^l) \wedge \phi_0^l$$

is an $Ad(\mathfrak{g})$ -equivariant map: $Ad(g)\lambda = \lambda Ad(g)$ for all $g \in Ad(\mathfrak{g})$. By Proposition 5.4 the image $*L_{\phi_0^l}(\Lambda_{(\phi_0^l)_-}^{n-l-1}) = \Lambda^1(\mathfrak{g})$, and apply Proposition 5.4 again, we conclude that the image of λ is $\Lambda_{(\phi_0^l)_-}^{n-l-1}$. Thus λ is an $Ad(\mathfrak{g})$ -equivariant endomorphism of $\Lambda_{(\phi_0^l)_-}^{n-l-1}$. The same argument as in the proof of Lemma 5.3 using Schur's Lemma implies that λ is a multiple of the identity map. This proves Lemma 5.16. \square

Let us continue the proof of Proposition 5.15. Suppose that $\beta \in \mathcal{H}_{\phi_-}^{n-l-1}$. Set

$$(5.31) \quad \alpha := *(\beta \wedge \phi^l) \in \Omega^1(M^n).$$

Using $d*\beta = 0$, $d\phi^l = 0$ and taking into account Lemma 5.16 we get

$$d\alpha \wedge \phi^l = d(\alpha \wedge \phi^l) = d(*(\beta \wedge \phi) \wedge \phi) = c \cdot d*\beta = 0,$$

which implies that $d\alpha \in \Omega_{\phi_+^l}^2(M^n)$. Hence

$$(5.32) \quad d_- \alpha = 0.$$

Next we note that

$$(5.33) \quad d * \alpha = d(\beta \wedge \phi^l) = d\beta \wedge \phi = 0$$

since $d_- \beta = 0$.

By (5.32) and (5.33) α is ϕ_-^l -harmonic. This completes the proof of Proposition 5.15. \square

Lemma 5.17. *Let M^n be a connected compact $Aut(\mathfrak{g})$ -manifold provided with a l -form ϕ^l of type ϕ_0^l . Then*

1. *There is a monomorphism $i : H^1(M^n) \rightarrow \mathcal{H}_{\phi_-^l}^1(M^n)$. This monomorphism is an isomorphism if $\phi_0^l = \omega_{\mathfrak{g}}^3$.*
2. *If $\delta\phi^l = 0$, then $\mathcal{H}_{\phi_-^l}^{n-l}(M^n) = 1$.*

Proof. Let α be a harmonic 1-form on M^n . Then α is also a ϕ_-^l -harmonic form. Thus $i : H^1(M^n) \rightarrow \mathcal{H}_{\phi_-^l}^1(M^n)$ is a monomorphism. If $\phi_0^l = \omega_{\mathfrak{g}}$ then i is an isomorphism, since $H^1(M^n) = H_{(\omega_{\mathfrak{g}})_-}^1$ as a direct consequence of Corollary 5.6, taking into account Proposition 5.14. This proves the first assertion of Lemma 5.17.

By Proposition 5.4 we can write $\Omega_{\phi_-^l}^{n-l}(M^n) = \langle *\phi^l \rangle_{\mathbb{R}}$, hence

$$(5.34) \quad \mathcal{H}_{\phi_-^l}^{n-l}(M^n) = \{f\phi \in \Lambda_{\phi_-^l}^{n-l} \mid d(f * \phi^l) = 0\}.$$

Since ϕ^l is harmonic, (5.34) holds if and only if f is constant. This completes the proof of Lemma 5.17. \square

Remark 5.18. The construction of our differential complexes is similar to the construction of differential complexes on a 7-manifold with an “integrable” G_2 -structure by Fernandez and Ugarte in [7]. Their consideration is based on Reyes work [30], where Reyes also considered differential complexes associated to certain G -structures, using some ideas in [32]. Many special properties of our complexes are related to the cohomological structure of a simple compact Lie algebra \mathfrak{g} . A subcomplex $\Omega_{(*\omega_{\mathfrak{g}})_+}^2$ will be shown to play a roll in the geometry of strongly associative submanifolds of dimension 3, see Proposition 6.6.

6. SPECIAL SUBMANIFOLDS IN $Aut(\mathfrak{g})$ -MANIFOLDS

In this section we also assume that \mathfrak{g} is a compact simple Lie algebra and M^n is a smooth manifold provided with a 3-form of type $\omega_{\mathfrak{g}}$, where \mathfrak{g} is a compact simple Lie algebra. The Lie bracket on \mathfrak{g} extends smoothly to a cross-product $TM \times TM \rightarrow TM$, which we denote by $[\cdot, \cdot]_{\mathfrak{g}}$. We study a natural class of submanifolds in $Aut(\mathfrak{g})$ -manifolds, which generalize the notion of Lie subgroups in a Lie group G , and show their algebraic and geometric

rigidity, see Proposition 6.2, Proposition 6.4, Proposition 6.6, Proposition 6.8, Remark 6.9.

Definition 6.1. Let M^n be a manifold provided with a 3-form ω^3 of type $\omega_{\mathfrak{g}}$. A submanifold $N^k \subset M^n$ is called *associative*, if the tangent space TN^k is closed under the bracket $[\cdot, \cdot]_{\mathfrak{g}}$. A submanifold $N^k \subset M^n$ is called *coassociative*, if its normal bundle $T^\perp N^k$ is closed under the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$. An associative (resp. coassociative) submanifold $N^k \subset M^n$ is called *strongly associative* (resp. *strongly coassociative*), if the restriction of ω^3 to TN^k (resp. to $T^\perp N^k$) is multi-symplectic.

The following Proposition shows the algebraic rigidity of strongly associative and strongly coassociative submanifolds. It is a direct consequence of Lemma 2.1, so we omit its proof.

Proposition 6.2. *Assume that N^k is a strongly associative submanifold in M^n . Then all the tangent spaces $(T_x N^k, [\cdot, \cdot]_{\mathfrak{g}})$ are isomorphic to a compact semisimple Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Assume that N^k is a strongly coassociative submanifold in M^n . Then all the normal spaces $(T_x^\perp N^k, [\cdot, \cdot]_{\mathfrak{g}})$ are isomorphic to a compact semisimple Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$.*

We call the semisimple Lie algebra \mathfrak{h} described in Proposition 6.2 the *Lie type* of a strongly associative (resp. strongly coassociative) submanifold N^k .

The following Lemma is well-known, so we will omit its proof.

Lemma 6.3. *Assume that N^k is an associative submanifold in a manifold M^n provided with a closed 3-form ω^3 . Then the distribution $\ker \omega^3|_{N^k}$ is integrable.*

Examples from torsion-free $\text{Aut}(\mathfrak{g})$ -manifolds shows that associative submanifolds where the Lie bracket on the tangent space vanishes can have arbitrary sharp, unless they are of maximal dimension. In contrary the geometry of strongly associative (or strongly coassociative) submanifolds is partially controlled by the Cartan 3-form ω^3 .

For any oriented submanifold N^k in M^n denote by $T_x \vec{N}^k$ the unit k -vector associated with the tangent subspace $T_x N^k \subset T_x M^n$.

Proposition 6.4. *1. Assume that N^3 is a strongly associative 3-submanifold in a manifold M^n provided with a 3-form ω^3 of type $\omega_{\mathfrak{g}}$. Then the mean curvature H of N^3 satisfies the following equation for any $x \in N^3$*

$$(6.1) \quad H(x)^* = c \cdot T_x \vec{N}^3 \rfloor d\omega^3(x),$$

where $H(x)^*$ denotes the covector in $T_x^* M$ dual to $H(x) \in T_x M$ with respect to the Riemannian metric $K_{\mathfrak{g}}(x)$, and c is a positive constant depending only on the Lie type of N^3 .

2. Assume that N^{n-3} is strongly coassociative submanifold of codimension 3 in a manifold M^n provided with a 3-form ω^3 of type $\omega_{\mathfrak{g}}$. Then the mean curvature H of N^{n-3} satisfies the following equation for any $x \in N^{n-3}$

$$(6.2) \quad H(x)^* = c \cdot T_x \vec{N}^{n-3} \rfloor d(*\omega^3)(x),$$

where $H(x)^*$ denotes the covector in T_x^*M dual to $H(x) \in T_xM$ with respect to the Riemannian metric $K_g(x)$, and c is a positive constant depending only of the Lie type of N^{n-3} .

Proof. Let us choose $c^{-1} = \omega_g(T_x \vec{N}^3)$ for strongly associative submanifold N^3 , $x \in N^3$, and let us choose $c^{-1} = *(\omega_g)(T_x \vec{N}^{n-3})$ for strongly coassociative submanifold N^{n-3} , $x \in N^{n-3}$. By Theorem 3.1 in [19] c is a nonzero critical value of the function $f_{\omega_g}(x)$ (resp. $f_{*\omega_g}(x)$) defined on each Grassmannian $Gr_3^+(T_x M^n)$ of oriented 3-vectors, $x \in N^3$, (resp. on $Gr_{n-3}^+(T_x M^n)$, $x \in N^{n-3}$) by setting $f_{\omega}(u) := \omega_g(x)(\vec{u})$ (resp. $f_{\omega}(u) := *\omega_g(x)(\vec{u})$).

Now Proposition 6.4 follows directly from [19, Lemma 1.1], where we computed the first variation formula for a Riemannian submanifold $N^k \subset M$ satisfying the condition that there is a differential form ω on M such that $\omega|_{N^k} = \text{vol}|_{N^k}$ and moreover the value $\omega(x)(T_x N^k)$ is a nonzero critical value of the function $f_{\omega}(x)$ defined on the Grassmannian $Gr_k^+(T_x M)$ by setting $f_{\omega}(u) := \omega(\vec{u})$. For a strongly associative submanifold N^3 Lemma 1.1 in [19] yields

Lemma 6.5. (cf [19, Lemma 1.1]) *For any point $x \in N^3$ and for any normal vector $X \in T_x^\perp N^3$ we have*

$$(6.3) \quad \langle -H(x), X \rangle = c \cdot d\omega^3(X \wedge T_x \vec{N}^3).$$

Clearly (6.4) is equivalent to (6.3). In the same way we prove (6.2). \square

Strongly associative or coassociative submanifolds of (co) dimension 3 thus satisfy the equation in Proposition 6.4, which is a first order perturbation of the second order equation describing minimal submanifolds. On the other hand, as in [31, Proposition 2.3] we can also define strongly associative submanifolds (resp. strongly coassociative manifolds) as integral submanifolds of some differential system on M .

Proposition 6.6. *A 3-submanifold $N^3 \subset M^n$ is strongly associative, if and only if the restriction of any 3-form on M with value in the subbundle $d_g(\Omega_{(*\omega_g)_+}^2)$ to N^3 vanishes. A submanifold $N^{n-3} \subset M^n$ is strongly coassociative, if the restriction of any $(n-3)$ -form on M with value in the subbundle $*d_g(\Omega_{(*\omega_g)_+}^2)$ to N^{n-3} vanishes.*

Proof. The first assertion of Proposition 6.6 is a direct consequence of [31, Proposition 2.3] and Formula (4.12). The second assertion is a consequence of [31, Proposition 2.3], Formula (4.12) and the following identity

$$\rho_*(\tau)(* \omega_g) = *(\rho_*(\tau) \omega_g)$$

for any $\tau \in \mathfrak{so}(\mathfrak{g}) \cong \Lambda^2(\mathfrak{g})$. \square

Propositions 6.4 and 6.6 show that strongly associative submanifolds in dimension 3 and strongly coassociative submanifolds in co-dimension 3 behave like calibrated submanifolds.

Proposition 6.4 can be generalized for other strongly (co)associative submanifolds N^k , when $k \geq 4$, under some additional conditions on Lie type of N^k . We say that a compact Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is *critical with respect to an $Ad(\mathfrak{g})$ -invariant k -form ϕ_0^k on \mathfrak{g}* , if $\vec{\mathfrak{h}}$ is a critical point corresponding to a nonzero critical value of the function $f_{\phi_0^k}$ defined on $Gr_k^+(\mathfrak{g})$ as we defined above for $\phi_0^k = \omega_{\mathfrak{g}}$.

Example 6.7. In [19] we showed that the canonical embedded Lie subalgebra $\mathfrak{su}(n) \rightarrow \mathfrak{su}(m)$, $n \leq m$, is critical with respect to an $Ad(\mathfrak{g})$ -invariant form ϕ^{n^2-1} which is the wedge product of elementary $Ad(\mathfrak{g})$ -invariant $(2k-1)$ -forms θ^{2k-1} defined on $\mathfrak{su}(m)$ as follows

$$\begin{aligned} \theta^{2k-1}(X_1, \dots, X_{2k-1}) &= Re \sum_{\sigma \in \Sigma_{2k-1}} \varepsilon_{\sigma} Tr(X_{\sigma(1)} \cdots X_{\sigma(2k-1)}) \text{ if } k \text{ is even,} \\ \theta^{2k-1}(X_1, \dots, X_{2k-1}) &= Im \sum_{\sigma \in \Sigma_{2k-1}} \varepsilon_{\sigma} Tr(X_{\sigma(1)} \cdots X_{\sigma(2k-1)}) \text{ if } k \text{ is odd,} \end{aligned}$$

where $X_i \in \mathfrak{su}(m)$ and the multiplication is the usual matrix multiplication.

In [19] and [20] using different methods we proved that the canonical embedded group $SU(n) \rightarrow SU(m)$ is stable minimal with respect to the bi-invariant Riemannian metric on $SU(m)$.

Proposition 6.8. *Assume that a Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is a critical with respect to an $Aut^+(\mathfrak{g})$ -invariant k -form ϕ_0^k on \mathfrak{g} . Then any associative submanifold N^k of Lie type \mathfrak{h} satisfies*

$$(6.4) \quad H(x)^* = T_x \vec{N}^k \rfloor d\phi^k(x),$$

where $H(x)^*$ denotes the covector in T_x^*M dual to $H(x) \in T_x M$ with respect to the Riemannian metric $K_{\mathfrak{g}}(x)$, and ϕ^k is the extension of ϕ_0^k on M^n . Moreover N^k is orientable. A similar statement also holds for strongly coassociative submanifolds $N^{n-k} \subset M^n$.

Proof. We can assume that 1 is a critical value of the function $f_{\phi_0^k}$ defined on $Gr_k^+(\mathfrak{g})$ and \mathfrak{h} is a critical point corresponding to this value. Now applying [19, Lemma 1.1] to N^k we obtain Proposition 6.8 immediately. (The orientability of N^k is a consequence of the fact that the restriction of ϕ^k to N^k is a nonzero form of top degree). \square

Remark 6.9. As a consequence of a result by Robles [31, Lemma 5.1], the differential ideal generated by differential forms taking value in $d_{\mathfrak{g}}(\Omega_{(\omega_{\mathfrak{g}})_+}^2)$ (resp. in $*d_{\mathfrak{g}}(\Omega_{(\omega_{\mathfrak{g}})_+}^2)$) is differentially closed, if M^n is torsion-free. Robles also shows that in the torsion-free case the differential system corresponding to coassociative submanifolds of codimension 3 is not in involution. It is possible that the considered differential system is not minimal in the sense that there is a smaller differential system having the same integral submanifolds, see [31, Remark 5.2].

7. FINAL REMARKS

1. The richness of geometry of manifolds equipped with a simple Cartan 3-form demonstrated in our note shows that these manifolds could be considered as natural generalizations of the notion of compact simple Lie groups in the category of Riemannian manifolds. There are many interesting 1-order differential operators on such manifolds, which we could exploit further to understand the geometry of underlying manifolds. One of the operators we have in our mind is the elliptic self-adjoint operator $d + d^* + \lambda(d_{\mathfrak{g}} + \delta_{\mathfrak{g}})$, $\lambda \in \mathbb{R}$, acting on $\Omega^*(M)$. It would be interesting to develop the theory further for manifolds of special algebraic type, especially to find topological constraints of manifolds with harmonic ϕ^l -forms.

2. We have discussed in this note only first order invariants of considered manifolds and do not discuss the curvature of underlying Riemannian metric as well as relations between first order invariants and second order invariants of these manifolds.

3. It would be interesting to study Lie group actions and moments maps on manifolds with a closed form of type ϕ^l , see also [23].

4. Find sufficient and necessary conditions to ensure the local existence of a class (co) associative submanifolds.

8. APPENDIX. RIEMANNIAN MANIFOLDS EQUIPPED WITH A PARALLEL 3-FORM

In this Appendix we describe simply-connected complete Riemannian manifolds provided with a parallel 3-form. Let us first recall the following basis definitions.

A 3-form ω^3 on $\mathbb{R}^6 = \mathbb{C}^3$ is called a *Special Lagrangian 3-form* (or *SL-form*), if ω can be written as $\omega^3 = \text{Re}(dz^1 \wedge dz^2 \wedge dz^3)$.

A 3-form ω^3 on $\mathbb{R}^7 = \text{Im } \mathbb{O}$ is called of *G_2 -type*, if ω^3 is on the $GL(\mathbb{R}^7)$ -orbit of the form $\omega_0^3(X, Y, Z) := \langle XY, Z \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product on the octonion algebra \mathbb{O} .

A 3-form ω^3 on \mathbb{R}^{2n+1} is called of *product type of maximal rank*, if ω^3 can be written as $\omega^3 = dz \wedge (\sum_{i=1}^n dx^i \wedge dy^i)$.

A 3-form ω^3 on \mathbb{R}^n is called a *compact simple Cartan form*, if $\omega^3 = \omega_{\mathfrak{g}}$ for some compact simple Lie algebra \mathfrak{g} .

Theorem 8.1. *Let M^n be a connected simply connected complete Riemannian manifold provided with a parallel 3-form ω^3 . Then (M^n, ω^3) is a direct product of basis Riemannian manifolds M_i provided with a parallel k -form ω_i of the following types.*

- 1) A Calabi-Yau 6-manifold M^6 provided with a SL 3-form.
- 2) A (torsion-free) G_2 -manifold M^7 provided with a 3-form of G_2 -type.
- 3) A compact simple Lie group or its noncompact dual with the associated Cartan 3-form.
- 4) A Kähler manifold M^{2n} with a Kähler form ω^2 .

5) A Euclidean space (\mathbb{R}^k, ϕ^3) provided with a parallel multi-symplectic 3-form ϕ^3 .

6) A Riemannian manifold N^l with the zero 3-form.

The 3-form ω^3 is a sum of the SL -3-forms, G_2 -forms, a multiple by a nonzero constant of the Cartan 3-forms, and 3-forms of product type of maximal rank, whose precise description will be given in the proof below.

Proof. Let G be the holonomy group of a connected simply connected Riemannian manifold M^n provided with a parallel 3-form ω^3 . By the de Rham theorem, see e.g. [2, 10.43, chapter 10], M^n is a product of Riemannian manifolds $M_i, i \in I$, such that $G = \prod_{i \in I} G_i$ where G_i acts irreducibly on TM_i for $i \geq 1$ and M_0 is flat. Now choose an arbitrary point $x \in M^n$. Let L_{ω^3} be the map defined in (2.1). Since $\ker L_{\omega^3}$ is a G -module, we have the following decomposition

$$\ker L_{\omega^3} = \oplus_{i \in J} T_x M_i \oplus (\ker L_{\omega^3} \cap T_x M_0),$$

where J is some subset of I . Let N be a submanifold in M satisfying the following conditions:

1. $N \ni x$ and $T_x N = \ker L_{\omega^3}$,
2. N is invariant under the action of G .

Note that there is a unique submanifold $N \subset M$ satisfying the conditions 1 and 2. Let W be the orthogonal complement to $\ker L$. Denote by \mathbb{R}^p the subspace of the flat space M_0 which is tangent to W . Set

$$(8.1) \quad M_m^n := \prod_{i \in I \setminus J} M_i \times \mathbb{R}^p.$$

Then $M^n = N \times M_m^n$. Let $\pi_m : M^n \rightarrow M_m^n$ be the natural projection. Clearly $\omega^3 \in \pi^*(\Omega^3(M_m^n))$. Denote by $\bar{\omega}_i^3$ the restriction of ω^3 to M_i which enter in the decomposition (8.1). Since G preserves ω^3 , the subgroup G_i preserves the 3-form $\bar{\omega}_i^3$.

Lemma 8.2. *If $\bar{\omega}_i^3$ is not zero and G_i acts irreducibly on TM_i then $\bar{\omega}_i^3$ is either a SL 3-form, or a G_2 -form, or a multiple of a Cartan simple form.*

Proof. Note that G_i is a subgroup of the stabilizer of $\bar{\omega}_i^3$. First we inspect the list of all possible irreducible Riemannian holonomy groups in [2, table 1, chapter 10], using the fact that the groups $U(n), SU(n)$ (resp. $Sp(n), Sp(1)Sp(n)$) have no invariant nonzero 3-form on the space \mathbb{R}^{2n} , if $n \neq 3$ (resp. on the space \mathbb{R}^{4n} for all n), as well as $Spin(7)$ has no-invariant 3-form on \mathbb{R}^8 , to conclude that M_i must be a symmetric space of type I or IV, or a Calabi-Yau 6-manifold, or a G_2 -manifold. Using the table of Poincare polynomials for symmetric spaces of type 1 in [34] we obtain that the only symmetric spaces of type 1 and of dimension greater than equal 3 with non-trivial Betti number b_3 are compact simple Lie groups. This completes the proof of Lemma 8.2. \square

Now we assume that $\bar{\omega}_i^3 = 0$. We say that the 3-form ω^3 has *rank 2* on M_i , if for some $x \in M_i$ (and hence for all $x \in M_i$) the linear map

$L_{\omega^3}^2 : \Lambda^2 T_x M_i \rightarrow T_x^* M, v \wedge w \mapsto v \wedge w \rfloor \omega^3$, has nonzero image. We say that ω^3 has *rank 1* on M_i if the image $L_{\omega^3}^2$ is zero and for some $x \in M_i$ (and hence for all $x \in M_i$) the map $L_{\omega^3} : TM_i \rightarrow T^*M, v \mapsto v \rfloor \omega^3$ has nonzero image. If both the maps $L_{\omega^3}^2$ and L_{ω^3} have trivial image, then clearly ω^3 belongs to the space $\Lambda^3(T^*(M_1 \times \cdots \times M_k))$.

Lemma 8.3. *Assume that $\bar{\omega}_i^3 = 0$. Then the 3-form ω^3 has rank 2 on M_i and M_i is a Kähler manifold provided with a Kähler 2-form ω_i^2 . Moreover there is a subspace $\mathbb{R}_i \subset \mathbb{R}^p$ provided with a constant 1-form dx^i such that the restriction of ω^3 to $M_i \times \mathbb{R}_i$ is equal to $\lambda_i dx^i \wedge \omega_i^2$ for some nonzero constant λ_i .*

Proof. Assume that ω^3 has rank 1 on M_i . Then there is a vector $v \in T_x M_i$ and two vectors $u, w \in T_x M_m^n$ such that u, v are orthogonal to $T_x M_i$ and $\omega^3(v, u, w) = 1$. Since u, v are orthogonal to $T_x M_i$ the 1-form $u \wedge w \rfloor \omega^3$ is invariant under the action of G_i . Taking into account the irreducibility of the action of G_i on $T_x M_i$ we obtain the first assertion of Lemma 8.3. Thus ω^3 has rank 2 on M_i , if $\bar{\omega}_i^3 = 0$. It follows that there is a vector $v \in T_x M_m^n$ such that v is orthogonal to $T_x M_i$ and $\omega^3(v, u, w) = 1$ for some two vectors $u, w \in T_x M_i$. Repeating the previous argument, we conclude that $v \in T_x \mathbb{R}^p$ and G_i leaves $\Omega_i := (u \rfloor \omega^3)|_{T_x M_i}$ invariant. Since G_i acts irreducibly on $T_x M_i$, the 2-form Ω_i has maximal rank. We conclude that M_i is a Kähler manifold. Denote by Λ_i the bivector in $\Lambda^2 T_x M_i$ which is dual to Ω_i with respect to the Riemannian metric. Since the Kähler form on M_i is defined uniquely up to a nonzero scalar multiple, the two-vector Λ_i is defined uniquely up to a nonzero scalar multiple in the sense that Λ_i does not depend on the original vector u . Now set $\hat{u}_i := (\Lambda_i \rfloor \omega^3)|_{\mathbb{R}^p}$. Let $\bar{u}_i \in T_x \mathbb{R}^p$ be the vector dual to \hat{u}_i with respect to the given Riemannian metric. Then $\hat{u}_i(\bar{u}_i) = 1$. Clearly the restriction of ω^3 to $M_i \times \langle \bar{u}_i \rangle_{\mathbb{R}}$ is a 3-form of the product type of maximal rank. This completes the proof of Lemma 8.3. \square

Let us complete the proof of Theorem 8.1. Set

$$\omega_m^3 := \sum_i \pi_i^*(\bar{\omega}_i^3) + \pi_0^*(\omega_{|\mathbb{R}^p}^3),$$

where $i \in I \setminus J$ and $\pi_i : M \rightarrow M_i, \pi_0 : M \rightarrow \mathbb{R}^p$ are the natural projections. Clearly $\omega^3 - \omega_m^3$ is also invariant under the action of G . We note that the restriction of $\omega^3 - \omega_m^3$ to each M_i is zero, hence the rank of $\omega^3 - \omega_m^3$ to each M_i has rank 2 by Lemma 8.3. Next we observe that the restriction of $\omega^3 - \omega_m^3$ to \mathbb{R}^p vanishes and it has rank 1, since in the opposite case, using the same argument as in the proof of Lemma 8.3 we conclude that one of the space $M_i, i \in I \setminus J$, is flat, which contradicts our assumption.

Lemma 8.4. *We have $\omega^3 - \omega_m^3 - \sum \Omega^i \wedge \hat{u}_i = 0$.*

Proof. By our construction the 3-form $\omega^3 - \omega_m^3 - \sum_i \Omega^i \wedge \hat{u}_i$ belongs to the space $\pi_0^*(\Omega^3(\mathbb{R}^p))$. On the other hand, the restriction of $\omega^3 - \omega_m^3 - \sum \Omega^i \wedge \hat{u}_i$ to \mathbb{R}^p is zero. This proves Lemma 8.4. \square

Clearly Lemma 8.4 completes the proof of Theorem 8.1. \square

ACKNOWLEDGEMENT

This note is partially supported by grant of ASCR Nr IAA100190701. The author thanks Sasha Elashvili, Willem Graaf, Todor Milev, Dmitri Panushev, Colleen Robles, Andrew Swann, Jiří Vanžura for their helpful remarks, and Xiaobo Liu, Yoshihiro Ohnita, Hiroyuki Tasaki for sending their reprints. A part of this note has been written during the author stay at the ASSMS, GCU, Lahore-Pakistan. She thanks ASSMS for their hospitality and financial support.

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